MIDTERM - MON, MARCH 28

ANNOTATED BIBLIOGRAPHY - MON, APR 4

- AT LEAST 4 REFERENCES
- FOR EACH: - BULLET POINT SUMMARY OF RELEVANT POINTS

TODAY: ATOM-LIGHT INTERACTION (INTRO)
(FOOT 7.1)

WARM-UP (REVIEW OF HW2 #3)

He $1s^2$ CFA

GIVEN: $V_{cf}(r) = \frac{-e^2}{4\pi\epsilon_0 r}\left[1 + e^{-4r/a_0}(1+2r/a_0)\right]$

Approximate $V_{cf}(r)$ in the form $\frac{(?)}{r} + \text{const}$ in the limits:

a) $r/a_0 \gg 1$ $e^{-4r/a_0} \to 0$

$V_{cf}(r) \to \boxed{\frac{-e^2}{4\pi\epsilon_0 r}}$ (charge screening by $1e^-$)

b) $\frac{r}{a_0} \ll 1$ (NOTE: $e^x \approx 1+x$ FOR $x \ll 1$)

$$e^{-4r/a_0} \approx 1 - 4r/a_0$$

$$e^{-4r/a_0}(1+2r/a_0) \approx (1-4r/a_0)(1+2r/a_0) \approx 1 - 2r/a_0$$

$$V_{cf}(r) \to \frac{-e^2}{4\pi\epsilon_0 r}(1+1-2r/a_0) = \boxed{\frac{e^2}{4\pi\epsilon_0}\left(\frac{-2}{r} + \frac{2}{a_0}\right)}$$

$\vec{E} = -\vec{\nabla} V$, so $\vec{E}$ is same as bare nucleus but V is higher because of other electron

ATOM-LIGHT INTERACTION

OVERVIEW: FUNDAMENTAL PROCESSES

- ABSORPTION
- STIM. EMISSION

} CLASSICAL EM FIELD ✓

- SPONTANEOUS EMISSION ← QUANTIZED EM FIELD

LIGHT: EM RADIATION

- USE SCALAR + VECTOR POTENTIALS

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

- $\vec{E}+\vec{B}$ UNCHANGED BY GAUGE TRANSFORMATION:

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi$$

$$\phi' = \phi - \partial\chi/\partial t$$

(χ = ANY SCALAR FUNCTION)

- CHARGED PARTICLE IN EM FIELD:

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$$

ELECTRON: $q = -e$

CONSIDER HYDROGEN IN PLANE WAVE OF LIGHT

$\vec{A} = ?$ $\phi = ?$

- DEPENDS ON GAUGE
- USE COULOMB GAUGE: $\nabla\cdot A = 0$

(NOTE: GIVEN ANY $\vec{A},\phi$, $0 = \nabla\cdot A' = \nabla\cdot(A + \nabla\chi) = \nabla\cdot A + \nabla^2\chi$

$\rightarrow \nabla^2\chi = -\nabla\cdot A$ = GIVEN (POISSONS' EQN)

$\rightarrow$ SOLUTION EXISTS $\rightarrow$ CAN ALWAYS USE COULOMB GAUGE)

- GAUSS'S LAW: $\rho/\varepsilon_0 = \nabla\cdot E = \nabla\cdot(-\nabla\phi - \partial_t A)$

$$= -\nabla\phi - \partial_t\underbrace{(\nabla\cdot A)}_{0} = -\nabla\phi$$

$\rightarrow$ $\phi(\vec{r},t)$ is given by instantaneous charge density

FOR ELECTRON IN HYDROGEN,

$$\phi(\vec{r},t) = \frac{Ze}{4\pi\varepsilon_0 r}$$

VECTOR POTENTIAL

$\vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}$

CHARGES ($-\vec{\nabla}\phi$) RADIATION ($\partial_t \vec{A}$)

PLANE WAVE:

$$\vec{A}(\vec{r},t) = \vec{A}_0 e^{i(k\cdot r - \omega t)} + c.c.$$

COULOMB GAUGE: $\nabla\cdot A = 0 \Rightarrow k\cdot A_0 = 0$

$$\begin{cases} E_R(\vec{r},t) = \underbrace{\left(-i\omega\vec{A}_0\right)}_{\vec{E}_0} e^{i(k\cdot r-\omega t)} + c.c. \\ \vec{B}_R(r,t) = \underbrace{\left(-i\,\vec{k}\times\vec{A}_0\right)}_{\vec{B}_0} e^{i(k\cdot r-\omega t)} + c.c. \end{cases}$$

HYDROGEN ATOM + LIGHT (NEGLECTING SPIN):

$$H = \frac{1}{2m_e}\left(\vec{p} + e\vec{A}\right)^2 - \frac{Ze^2}{4\pi\epsilon_0 r}$$

$$= \underbrace{\left(\frac{\vec{p}^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0 r}\right)}_{H_0} + \underbrace{\frac{e}{2m_e}\left(p\cdot A + A\cdot p\right) + \frac{e^2}{2m_e}A^2}_{H'}$$

- NEGLECT A^2 TERM EXCEPT FOR HIGH INTENSITY LIGHT ($E_0 \sim$ NUCLEAR $\vec{E}$ FIELD)

- IN COULOMB GAUGE, $\vec{p}\cdot\vec{A} = \vec{A}\cdot\vec{p}$

$$\vec{p}\cdot(\vec{A}\psi) = -i\hbar\nabla\cdot(\vec{A}\psi)$$

$$= -i\hbar\underbrace{(\nabla\cdot A)}_{0}\psi - i\hbar\vec{A}\cdot\nabla\psi = (\vec{A}\cdot p)\psi$$

$$H' \approx \frac{e}{m_e}\vec{A}\cdot\vec{p}$$

TIME-DEPENDENT PERT. THEORY

$$H(t) = H_0 + H'(t)$$

$$|\psi(t)\rangle = c_1(t)e^{-i\omega_1 t}|1\rangle + c_2(t)e^{-i\omega_2 t}|2\rangle + \cdots = \sum_k c_k e^{-i\omega_k t}|k\rangle$$

$$(\omega_i = E_i/\hbar)$$

PLUG INTO TDSE $(i\hbar\partial_t|\psi\rangle = H|\psi\rangle)$:

$$\dot{c}_b = \frac{1}{i\hbar}\sum_k H'_{bk} c_k e^{i\omega_{bk}t}$$

WHERE $\omega_{bk} = \omega_b - \omega_k$

$$H'_{bk} = \langle b|H'(t)|k\rangle$$

INITIAL CONDITION: $c_a(0) = 1$, $c_{k\neq a}(0) = 0$

FIRST ORDER: $c_a(t) \approx 1$, $|c_{k\neq a}|^2 \ll 1$

$$\dot{c}_b^{(1)} = \frac{1}{i\hbar} H'_{ba} e^{i\omega_{ba}t}$$

$$c_b^{(1)}(t) = \frac{1}{i\hbar}\int_0^t H'_{ba}(t') e^{i\omega_{ba}t'} dt'$$

PERTURBATION:

$$H'(t) = \frac{e}{m_e}\vec{A}(t)\cdot\vec{p} = \frac{e}{m_e}\left[\vec{A}_0 e^{i(k\cdot r - \omega t)} + \vec{A}_0^* e^{i(-k\cdot r + \omega t)}\right]\cdot\vec{p}$$

$$\equiv v^\dagger e^{-i\omega t} + v e^{i\omega t}$$

$$\begin{cases} v = \dfrac{e}{m_e} e^{-i\vec{k}\cdot\vec{r}}\vec{A}_0^*\cdot\vec{P} \\[2ex] v^\dagger = \dfrac{e}{m_e}\vec{p}\cdot(\vec{A}_0 e^{ik\cdot r}) = \dfrac{e}{m_e} e^{ik\cdot r}\vec{A}_0\cdot\vec{P} \end{cases}$$

(COULOMB GAUGE $p\cdot A = p\cdot A$)

ABSORPTION & STIMULATED EMISSION

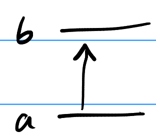
$$c_b^{(1)}(t) = \frac{1}{i\hbar}\int_0^t H'_{ba}(t')\, e^{i\omega_{ba}t'}\,dt'$$

$$= \frac{1}{i\hbar}\int_0^t \left(V_{ba}^{+} e^{-i\omega t'} + V_{ba}\, e^{i\omega t'} \right) e^{i\omega_{ba}t'}\,dt'$$

$$= \frac{1}{i\hbar}\left\{ V_{ba}^{+}\, \frac{e^{i(\omega_{ba}-\omega)t}-1}{i(\omega_{ba}-\omega)} + V_{ba}\, \frac{e^{i(\omega_{ba}+\omega)t}-1}{i(\omega_{ba}+\omega)} \right\}$$

(LET $\omega > 0$)

ABSORPTION: $E_b > E_a$ $\quad (\rightarrow \omega_{ba} > 0)$

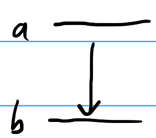


- FIRST TERM DOMINATES WHEN $\omega \approx \omega_{ba}$

  i.e. $\hbar\omega \approx E_b - E_a$

  RESONANT ABSORPTION

STIMULATED EMISSION: $E_a > E_b$ $\quad (\rightarrow \omega_{ba} < 0)$



- SECOND TERM DOMINATES WHEN $\omega \approx -\omega_{ba}$

  i.e. $\hbar\omega \approx E_a - E_b$

  RESONANT EMISSION

TODAY: - FINISH TDPT INTRO

- MULTIPOLE EXPANSION OF $A\cdot p$ INTERACTION

MIDTERM MON 3/28: MAKE 1-PAGE SHEET OF NOTES

WARM-UP (REVIEW):

Cr gnd state $[Ar]\,3d^5 4s^1$

GIVEN: the $3d^5$ electrons are in a ^6S state

FIND:

a) TOTAL L, S, J NUMBERS FOR GND STATE

$3d^5$: $2S+1=6 \rightarrow S=5/2$; $4s^1$: $S=1/2$

ADD: $S_{TOTAL} = 5/2 \pm 1/2$

HUND: MAX $S \rightarrow S = 5/2 + 1/2 = 3$

$2 \cdot 3 + 1 = 7$: ^7S

$L=0 \Rightarrow J=S=3 \rightarrow \boxed{^7S_3}$

b) LANDÉ g FACTOR g_J (ASSUME $g_S=2$, $g_L=1$)

$$g_J \approx \frac{3}{2} + \frac{S(S+1)-L(L+1)}{2J(J+1)} = \frac{3}{2} + \frac{1}{2} = \boxed{2}$$

- electron spins fully aligned

LAST TIME: HYDROGEN IN EM PLANE WAVE

- ATOM-LIGHT INTERACTION: $H' = \frac{e}{m_e} \vec{A} \cdot \vec{p}$

- PLANE WAVE: $\vec{A}(r,t) = \vec{A}_0 e^{i(k \cdot r - \omega t)} + \vec{A}_0^* e^{i(-k \cdot r + \omega t)}$
- COULOMB GAUGE: $\vec{A}_0 \cdot \vec{k} = 0$
- GIVES: $H'(t) = v^\dagger e^{-i\omega t} + v\, e^{i\omega t}$

WITH: $v = \frac{e}{m_e} e^{-i\vec{k}\cdot\vec{r}} \vec{A}_0^* \cdot \vec{p}$

$v^\dagger = \frac{e}{m_e} e^{i\vec{k}\cdot\vec{r}} \vec{A}_0 \cdot \vec{p}$

TDPT: $|\psi(t)\rangle = \sum_j c_j(t) e^{-i\omega_j t} |j\rangle$, $\omega_j = E_j / \hbar$

ASSUME $c_a(0) = 1$

$H'_{ba} = \langle b | H'(t) | a \rangle$

$\omega_{ba} = \omega_b - \omega_a$

$v_{ba} = \langle b | v | a \rangle$

$v^\dagger_{ba} = \langle b | v^\dagger | a \rangle$

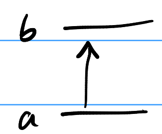
FIRST ORDER: $$C_b^{(1)}(t) = \frac{1}{i\hbar}\int_0^t H'_{ba}(t')e^{i\omega_{ba}t'}dt'$$

$$= \frac{1}{i\hbar}\int_0^t \left(\mathcal{V}_{ba}^{+} e^{-i\omega t'} + \mathcal{V}_{ba} e^{i\omega t'} \right) e^{i\omega_{ba}t'} dt'$$

$$= \frac{1}{i\hbar}\left\{ \mathcal{V}_{ba}^{+} \frac{e^{i(\omega_{ba}-\omega)t}-1}{i(\omega_{ba}-\omega)} + \mathcal{V}_{ba} \frac{e^{i(\omega_{ba}+\omega)t}-1}{i(\omega_{ba}+\omega)} \right\}$$

(LET $\omega > 0$)

ABSORPTION: $E_b > E_a$ $(\rightarrow \omega_{ba} > 0)$



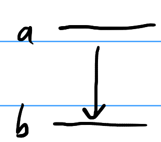
- FIRST TERM DOMINATES WHEN $\omega \approx \omega_{ba}$

  i.e. $\hbar\omega \approx E_b - E_a$

  RESONANT ABSORPTION

- REQUIRES $\mathcal{V}_{ba}^{+} \neq 0$

STIMULATED EMISSION: $E_a > E_b$ $(\rightarrow \omega_{ba} < 0)$



- SECOND TERM DOMINATES WHEN $\omega \approx -\omega_{ba}$

  i.e. $\hbar\omega \approx E_a - E_b$

  RESONANT EMISSION

- REQUIRES $\mathcal{V}_{ba} \neq 0$

EITHER WAY, RESONANT WHEN $\hbar\omega = |E_b - E_a|$

ELECTRIC DIPOLE APPROXIMATION

- $\vec{A}(\vec{r},t) = \vec{A}_0\, e^{-i\omega t}\, e^{ik\cdot r} + c.c.$

SIMPLIFY BY EXPANDING:

$$e^{ik\cdot r} \approx 1 + ik\cdot r + \frac{1}{2}(ik\cdot r)^2 + \cdots$$

- VALID WHEN $kr = 2\pi \frac{r}{\lambda} \ll 1$

e.g. $\lambda \sim 10^{-7}\,m$ (LIGHT)

$r \sim 10^{-10}\,m$ (BOHR RADIUS)

FIRST TERM: $e^{ik\cdot r} \approx 1$

$$\vec{A}(\vec{r},t) \approx \vec{A}_0 e^{-i\omega t} + c.c. = \vec{A}(0,t)$$

SCHRÖDINGER EQN:

$$\left[\frac{1}{2m}(\vec{p} + e\vec{A})^2 + V(r)\right]\psi(r,t) = i\hbar\,\partial_t\psi$$

UNITARY TRANSFORMATION:

$$\chi(\vec{r},t) = \vec{r}\cdot\vec{A}(0,t)$$

New WAVE FUNCTION:

$$\psi'(\vec{r},t) = e^{i\frac{e}{\hbar}\chi(\vec{r},t)}\,\psi(r,t) \qquad (e>0)$$

SCHRÖDINGER EQN. BECOMES

$$\Big[\underbrace{\frac{\vec{p}^2}{2m} + V(r)}_{H_0} - \underbrace{e\,\vec{r}\cdot\partial_t\vec{A}(0,t)}_{H'_{E1}}\Big]\psi' = i\hbar\,\partial_t\psi'$$

RECALL: $\vec{E} = -\partial_t\vec{A}$, DEFINE $\vec{d} = -e\vec{r}$

ELECTRIC DIPOLE INTERACTION: $\boxed{H'_{E1} = -\vec{E}(0,t)\cdot\vec{d}}$

PLANE WAVE:

$$\vec{E}(0,t) = \vec{E}_0 e^{-i\omega t} + \vec{E}_0^* e^{i\omega t}$$

$$H'_{E1} = (-\vec{E}_0 \cdot \vec{d}) e^{-i\omega t} + (-\vec{E}_0^* \cdot \vec{d}) e^{i\omega t}$$

$$= v^\dagger e^{-i\omega t} + v e^{i\omega t}$$

$$v = -\vec{E}_0^* \cdot (-e\vec{r}) = e\vec{E}_0^* \cdot \vec{r}$$

$$v^\dagger = e\vec{E}_0 \cdot \vec{r}$$

TRANSITION RATES DEPEND ON THE MATRIX ELEMENTS:

$$v^\dagger_{ba} = e\langle b|\vec{E}_0 \cdot \vec{r}|a\rangle$$

EXAMPLES:

1. LINEARLY POLARIZED LIGHT ALONG X:

LET $\vec{E}_0 = \frac{1}{2}\mathcal{E}\hat{x}$; $\mathcal{E}_0 \in \mathbb{R}$

a) $\vec{E}(0,t) = (\frac{1}{2}\mathcal{E}\hat{x})e^{-i\omega t} + (\frac{1}{2}\mathcal{E}\hat{x})e^{i\omega t} = \mathcal{E}\cos(\omega t)\,\hat{x}$

b) $\vec{E}_0 \cdot \vec{r} = \frac{1}{2}\mathcal{E}x$

c) $v^\dagger_{ba} = e\langle b|\frac{1}{2}\mathcal{E}x|a\rangle = \frac{1}{2}e\mathcal{E}\langle b|x|a\rangle$

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TODAY: SELECTION RULES IN HYDROGEN (FOOT 2.2)

- ANNOTATED BIBLIOGRAPHY DUE MON, APR 4

WARM-UP: FOR THE GIVEN $\vec{E}(0,t)$

a) DESCRIBE THE LIGHT POLARIZATION (LINEAR, CIRCULAR, ETC.)

b) FIND E_0 AND $\hat{\varepsilon}$ TO EXPRESS $\vec{E}(0,t)$ AS:

$$\vec{E}(0,t) = E_0 \hat{\varepsilon} e^{-i\omega t} + E_0 \hat{\varepsilon}^* e^{i\omega t}$$

WHERE $\hat{\varepsilon}^* \cdot \hat{\varepsilon} = 1$, $E_0 \in \mathbb{R}$

1. $\vec{E}(0,t) = A\cos(\omega t)\,\hat{e}_x$

$$= \frac{A}{2}\left(e^{-i\omega t} + e^{i\omega t}\right)\hat{e}_x \quad \rightarrow \quad \hat{e} = \hat{e}_x, \quad E_0 = A/2$$

LINEAR POLARIZATION

2. $\vec{E}(0,t) = A\,(\cos(\omega t), \sin(\omega t), 0)$

CIRCULAR POLARIZATION, CCW ABOUT z AXIS

$$\vec{E}(0,t) = A\left[\tfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\hat{e}_x + \tfrac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)\hat{e}_y\right]$$

$$= \frac{A}{2}\left[(\hat{e}_x + i\hat{e}_y)e^{-i\omega t} + (\hat{e}_x - i\hat{e}_y)e^{i\omega t}\right]$$

$$= \frac{\sqrt{2}}{2}A\left[\hat{e}_1 e^{-i\omega t} + \hat{e}_1^* e^{i\omega t}\right]$$

$$\boxed{E_0 = \frac{1}{\sqrt{2}}A}, \quad \hat{e} = \hat{e}_1 = \boxed{\frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)}$$

3. $\vec{E}(0,t) = A\,(\cos(\omega t), -\sin(\omega t), 0)$

CIRCULAR POLARIZATION, CW ABOUT z AXIS

SAME AS ABOVE, BUT $e_y \to -e_y$

$$E_0 = \frac{1}{\sqrt{2}}A, \quad \hat{e} = \hat{e}_{-1} = \boxed{\frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y)}$$

LAST TIME:

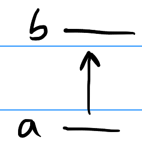
- ELECTRIC DIPOLE INTERACTION:

$$H'_{E1} = -\vec{E}(0,t)\cdot\vec{d}$$

- TDPT: FOR $H' = v^{\dagger} e^{-i\omega t} + v e^{i\omega t}$

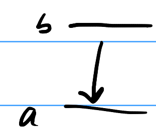
ABSORPTION $a \to b$:

$$|c_b^{(1)}(t)|^2 \propto |\langle b|v^{\dagger}|a\rangle|^2$$



STIM. EMISSION $b \to a$

$$|c_a^{(1)}(t)|^2 \propto |\langle a|v|b\rangle|^2$$



NOTE: $\langle b|v^{\dagger}|a\rangle^* = \langle a|v|b\rangle$

$$\Rightarrow |\langle b|v^{\dagger}|a\rangle|^2 = |\langle a|v|b\rangle|^2$$

TRANSITION MATRIX ELEMENT $\langle b|v^{\dagger}|a\rangle$

- FIND $v, v^{\dagger}$:

$$\vec{E}(0,t) = E_0\,\hat{\varepsilon}\,e^{-i\omega t} + E_0\,\hat{\varepsilon}^*\,e^{i\omega t}$$

$$\vec{d} = -e\vec{r}$$

$$H'_{E1}(t) = E_0\left(\hat{\varepsilon}\,e^{-i\omega t} + c.c.\right)\cdot e\vec{r}$$

$$= E_0 e(\hat{\varepsilon}\cdot\vec{r})\,e^{-i\omega t} + E_0 e(\hat{\varepsilon}^*\cdot\vec{r})\,e^{i\omega t}$$

$$\Rightarrow v^{\dagger} = E_0 e(\hat{\varepsilon}\cdot\vec{r})$$

$$\langle b|v^{\dagger}|a\rangle = E_0 e \underbrace{\langle b|\hat{\varepsilon}\cdot\vec{r}|a\rangle}$$

DIPOLE MATRIX ELEMENT: $\langle b|\hat{\varepsilon}\cdot\vec{r}|a\rangle$

HYDROGEN DIPOLE MATRIX ELEMENTS

$$\langle b | \vec{r}\cdot\hat{\varepsilon} | a\rangle = \int d^3r \; \psi^*_{n_b l_b m_b}(\vec{r})\,(\vec{r}\cdot\hat{\varepsilon})\,\psi_{n_a l_a m_a}(\vec{r})$$

$$= D_{ba}\, I_{ba}$$

$$D_{ba} = \int_0^\infty R_{n_b l_b}(r)\, r\, R_{n_a l_a}(r)\, r^2\, dr$$ "Radial integral"

$$I_{ba} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y^*_{l_b m_b}(\theta,\phi)(\hat{r}\cdot\hat{\varepsilon})\, Y_{l_a m_a}(\theta,\phi)$$ "Angular integral"

Angular integral $I_{ba} = 0$ except for specific conditions "SELECTION RULES"

SPHERICAL BASIS

$$\left.\begin{aligned} \hat{e}_1 &= -\tfrac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y) \\ \hat{e}_0 &= \hat{e}_z \\ \hat{e}_{-1} &= \tfrac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y) \end{aligned}\right\}$$

ORTHONORMAL

$$\hat{e}_q^* \cdot \hat{e}_{q'} = \delta_{qq'}$$

EXPAND $\hat{\varepsilon}$:

$$\hat{\varepsilon} = A_1\hat{e}_1 + A_0\hat{e}_0 + A_{-1}\hat{e}_{-1}$$

CCW LINEAR CCW (ALL RELATIVE TO z AXIS)

$$|A_1|^2 + |A_0|^2 + |A_{-1}|^2 = 1 \;\Rightarrow\; \hat{\varepsilon}^*\cdot\hat{\varepsilon} = 1$$

EXPAND $\hat{r}\cdot\hat{\varepsilon}$:

$$\hat{r} = \cos\phi\sin\theta\ \hat{e}_x + \sin\theta\sin\phi\ \hat{e}_y + \cos\theta\ \hat{e}_z$$

$$\hat{r}\cdot\hat{\varepsilon} \propto A_1 Y_{11} + A_0 Y_{10} + A_{-1} Y_{1,-1} \qquad \text{(HW)}$$

$$Y_{1,\pm1} \propto \sin\theta\ e^{\pm i\phi}$$

$$Y_{10} \propto \cos\theta$$

CONSIDER $\hat{\varepsilon} = \hat{e}_q$, $q = 0, \pm1$ i.e. $A_{q'} = \delta_{q'q}$

$$\hat{r}\cdot\hat{e}_q \propto Y_{1q}$$

$$I_{ba} \propto \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y^*_{\ell_b m_b}(\theta,\phi)\, Y_{1q}(\theta,\phi)\, Y_{\ell_a m_a}(\theta,\phi)$$

INTEGRAL ON ϕ: $Y_{\ell m} \propto \Theta_{\ell m}(\theta)\, e^{im\phi}$

$$\int_0^{2\pi} d\phi\ e^{i(-m_b+q+m_a)} = 0 \quad \text{UNLESS} \quad \boxed{m_b = m_a + q}$$

FULL ANGULAR INTEGRAL: USE A PROPERTY OF SPHERICAL HARMONICS

$$I_{ba} \propto \underbrace{\langle \ell_b m_b | 1q, \ell_a m_a \rangle}_{\text{CLEBSCH-GORDAN COEFFICIENT}}$$

ANGULAR MOMENTUM ADDITION:

$\ell_b = |\ell_a - 1|, \ldots, \ell_a + 1$

$m_b = m_a + q$ (ALREADY KNOW THAT)

PARITY (NEXT CLASS) $\Rightarrow \Delta \ell \neq 0$

ELECTRIC DIPOLE SELECTION RULES FOR HYDROGEN:

$$\Delta \ell = \pm 1$$

$$\Delta m = q = 0, \pm 1$$

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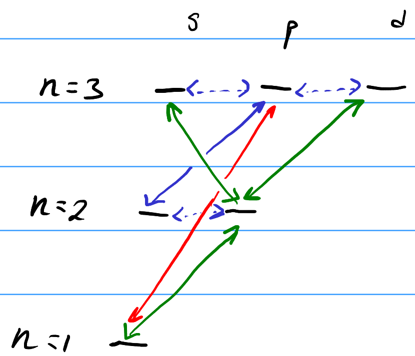
TODAY: SELECTION RULES PART 2

- PARITY
- MULTI-ELECTRON ATOMS

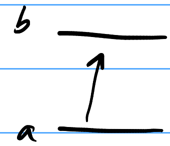
WARM UP: USING THE E1 SELECTION RULE

$$\Delta \ell = \pm 1$$

DRAW ARROWS SHOWING ALLOWED TRANSITIONS FOR HYDROGEN $n = 1, 2, 3$



REVIEW:



ABSORPTION

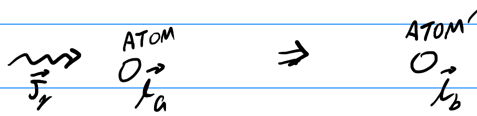
- TRANSITION PROB. $\propto |\langle b|\hat{\varepsilon}\cdot\vec{r}\,|a\rangle|^2$
  - POLARIZATION $\hat{\varepsilon} = \hat{e}_q$, $q = 0, \pm 1$ (spherical basis)

  ANGULAR INTEGRAL:

  $$\langle b|\hat{e}_q\cdot\vec{r}\,|a\rangle \propto \langle \ell_b m_b | 1q, \ell_a m_a\rangle$$

INTERPRETATION:

$$\vec{\ell}_b = \vec{\ell}_a + \vec{J}_\gamma \quad , \quad \vec{J}_\gamma = \text{PHOTON ANGULAR MOMENTUM}$$



$J_q = 1 \Rightarrow$ PHOTON IS SPIN 1

$q =$ SPIN PROJECTION OF PHOTON ALONG z AXIS

- ALLOWED ℓ_b, m_b FOLLOWS ANGULAR MOMENTUM ADDITION

  $\Rightarrow \Delta\ell = 0, \pm 1 \quad , \quad \Delta m = q$

– BUT $\Delta\ell \neq 0$! WHY NOT?

PARITY

- RECALL:

$$\int_{-\infty}^{\infty} f(x)\,dx = 0 \quad \text{if} \quad f(x) = -f(-x) \ (\text{ODD})$$

- EXAMPLE:

a) $\displaystyle\int_{-\infty}^{\infty} \underbrace{e^{-x^2}}_{\text{even}}\,dx \neq 0$

b) $\displaystyle\int_{-\infty}^{\infty} \underbrace{x\,e^{-x^2}}_{\text{odd}}\,dx = 0$

c) $\displaystyle\int_{-\infty}^{\infty} \underbrace{\underbrace{[x e^{-x^2}]}_{\text{ODD}}\underbrace{[x]}_{\text{ODD}}\underbrace{[e^{-x^2}]}_{\text{EVEN}}}_{\text{even}}\,dx \neq 0$

HYDROGEN ATOM WAVEFUNCTIONS ARE PARITY EIG. STATES:

$$P\,\psi_{n\ell m}(\vec{r}) = \psi_{n\ell m}(-\vec{r}) = (-1)^{\ell}\,\psi_{n\ell m}(\vec{r})$$

$$\langle b|\vec{r}\cdot\hat{\varepsilon}|a\rangle = \int d^3\vec{r}\;\psi_b^*(\vec{r})\,(\vec{r}\cdot\hat{\varepsilon})\,\psi_a(\vec{r}) \qquad \text{LET } \vec{r}\,' = -\vec{r}$$

$$= \int d^3\vec{r}\,'\;\psi_b^*(-\vec{r}\,')\,(-\vec{r}\,'\cdot\hat{\varepsilon})\,\psi_a(-\vec{r}\,')$$

$$= (-1)^{\ell_b+1+\ell_a}\langle b|\vec{r}\cdot\hat{\varepsilon}|a\rangle$$

$\Rightarrow \langle b|\vec{r}\cdot\hat{\varepsilon}|a\rangle = 0$ UNLESS $(-1)^{\ell_b+1+\ell_a} = 1$

$\Rightarrow \ell_b + 1 + \ell_a = \text{EVEN}$

$\Rightarrow \ell_b + \ell_a = \text{ODD}$

$\Rightarrow \ell_b \,\&\, \ell_a$ HAVE OPPOSITE PARITY

$\Rightarrow \ell_b \neq \ell_a$

SPIN

- WITHOUT SPIN-ORBIT COUPLING: $|\psi\rangle = |n \ell m_\ell m_s\rangle$

$$\langle \psi' | \hat{\varepsilon}\cdot\vec{r} | \psi \rangle = \langle n' \ell' m_\ell' | \hat{\varepsilon}\cdot\vec{r} | n \ell m_\ell \rangle \, \delta_{m_s' m_s}$$

$$\Rightarrow m_{s'} = m_s \;\ni\; \Delta m_s = 0$$

SPIN-ORBIT COUPLING:

$$|\psi\rangle = |n \ell j m_j\rangle$$

- COULD EXPAND $|j m_j\rangle$ USING C-G COEFFICIENTS...
  - BUT THERE'S A MORE GENERAL SOLUTION

WIGNER-ECKART THEOREM FOR VECTOR OPERATORS

THEOREM: LET $\vec{V}$ = ANY "VECTOR OPERATOR"

$\vec{J}$ = TOTAL ANGULAR MOMENTUM

γ = OTHER QUANTUM NUMBERS

$$\langle \gamma' J' M_J' | \hat{e}_q \cdot \vec{V} | \gamma J M_J \rangle = \underbrace{\langle J' M_J' | 1 q, J M_J \rangle}_{\text{CLEBSCH-GORDAN COEFFICIENT}} \underbrace{\langle \gamma' J' \| V \| \gamma J \rangle}_{\text{"REDUCED MATRIX ELEMENT"}}$$

- $\langle \gamma' J' \| V \| \gamma J \rangle$ DOESN'T DEPEND ON M_J', q, OR M_J

RELATIVISTIC HYDROGEN (FINE STRUCTURE) - FOOT 2.3.5

- $\gamma = (n, \ell)$

$$\langle n' \ell' j' m_j' | \hat{e}_q \cdot \vec{r} | n \ell j m_j \rangle = \langle j' m_j' | 1 q, j m_j \rangle \langle n' \ell' j' \| r \| n \ell j \rangle$$

ANGULAR MOMENTUM ADDITION

$$j' = |j-1|, \ldots, j+1$$

$$m_j' = m_j + q$$

SELECTION RULES:

$$\Delta j = 0, \pm 1$$

$$\Delta m_j = 0, \pm 1$$

HYPERFINE STRUCTURE

- SAME AS FOR J:

$$\Delta F = 0, \pm 1 \qquad (F = 0 \not\to 0)$$

$$\Delta M_F = 0, \pm 1$$

MULTI-ELECTRON ATOMS (FOOT 5.4)

- ELECTRIC DIPOLE INTERACTION

$$H'_{E1} = -\vec{E}(0,t) \cdot \vec{d}$$

$$\vec{d} = -e \sum_{i=1}^{N} \vec{r}_i$$

- Now $|c_b^{(1)}(t)|^2 \propto |\langle b| \vec{d} \cdot \hat{e}_q |a\rangle|^2$
- WIGNER-ECKART THEOREM:

$$\Delta J = 0, \pm 1 \qquad (J = 0 \not\to 0)$$

$$\Delta F = 0, \pm 1 \qquad (F = 0 \not\to 0)$$

$$\Delta M_F = 0, \pm 1 \qquad (M_F = 0 \not\to 0 \text{ WHEN } \Delta F = 0\text{, see CG Table})$$

- PARITY: $|a\rangle$ AND $|b\rangle$ HAVE OPPOSITE PARITY

APPROXIMATE RULES

CFA

- $\Delta \ell = \pm 1 \rightarrow$ ONE ELECTRON CHANGES BY $\Delta \ell_i = \pm 1$

  i.e. $1s^2 2s \rightarrow 1s^2 2p$

  OR $1s^2 2s \rightarrow 1s^2 np$, $n \geq 2$

LS COUPLING

- H'_{E1} ACTS ON $\vec{L}$, NOT $\vec{S}$:

$$\Delta S = 0$$

$$\Delta L = 0, \pm 1 \qquad (L = 0 \not\to 0)$$

EXAMPLE: Hg $6s^2\ {}^1S_0$ – $6s6p\ {}^3P_1$

	CHANGE?	E1 Allowed?
J	1	✓
ℓ	1	✓
L	1	✓
S	1	✗

- BUT Hg HAS STRONG S-O COUPLING THAT MIXES L, S

$$|\text{"}{}^3P_1\text{"}\rangle = \alpha|{}^3P_1\rangle + \beta|{}^1P_1\rangle$$

- ${}^1S_0 \to {}^1P_1$ IS ALLOWED

$\Rightarrow$ ${}^1S_0 \to$ "${}^3P_1$" IS WEAKLY ALLOWED

- MATRIX EL. IS ~10X WEAKER THAN ${}^1S_0 \to$ "${}^1P_1$"

PHY 446 SPRING 2022 LECTURE 19 WED, APR. 6

- HW 4 DUE WED APR 13
- PAPER DRAFT (HALF) DUE MON APR 18

TODAY: 2-LEVEL ATOM

RABI OSCILLATION

WARM-UP: HYDROGEN E1 TRANSITIONS

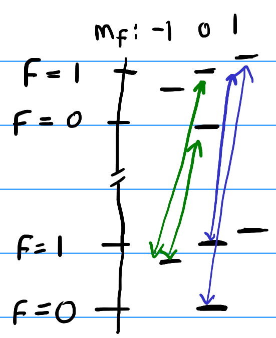
a) LIST SELECTION RULES FOR: n, ℓ, S, J, F, M_F

b) DRAW ARROWS FOR ALLOWED TRANSITIONS

GIVEN:
- STATIC $\vec{B}$ IN z DIRECTION
- CIRCULARLY POLARIZED LIGHT: $\hat{\varepsilon} = \hat{e}_1 = \frac{-1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$ "σ^+"

$2p$ $^2P_{1/2}$

$1s$ $^2S_{1/2}$



$\Delta J = 0, \pm 1$

$\Delta F = 0, \pm 1 \quad (0 \not\to 0)$

$\Delta M_F = q = 1 \quad (0 \not\to 0$ WHEN $\Delta F = 0)$

$\Delta \ell = \pm 1$

$\Delta S = 0$

TWO-LEVEL MODEL

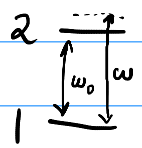
- QUESTION: APPLY LIGHT TO ATOM
  WHAT IS $|\psi(t)\rangle$?
- IMPORTANCE:
  - INDEX OF REFRACTION OF A GAS
  - OPTICAL TRAPPING OF ATOMS/PARTICLES

  DEPENDS ON INDUCED DIPOLE MOMENT

$$\langle \vec{d}(t)\rangle = \langle \psi(t)|\vec{d}|\psi(t)\rangle$$

TWO-LEVEL APPROX:

- LIGHT ($\omega > 0$) NEAR RESONANCE WITH $|1\rangle \to |2\rangle$ TRANSITION
- NEGLECT OTHER STATES



$$\omega \approx \omega_2 - \omega_1 \equiv \omega_0$$

$$\omega_2 = E_2/\hbar$$

$$\omega_1 = E_1/\hbar$$

TIME-DEPENDENT SCHRÖDINGER EQN.

$$H = H_0 + H'(t)$$

$$H_0|1\rangle = E_1|1\rangle, \quad H_0|2\rangle = E_2|2\rangle$$

$$H' = v^{+} e^{-i\omega t} + v e^{i\omega t}$$

E1 : $\vec{E}(0,t) = \frac{1}{2}E_0\,\hat{\varepsilon}\,e^{-i\omega t} + \frac{1}{2}E_0\hat{\varepsilon}^{*}e^{i\omega t} = Re\left[E_0\,\hat{\varepsilon}\,e^{-i\omega t}\right]$

$$H'_{E1} = -\vec{E}(0,t)\cdot\vec{d} = \underbrace{\left(-\tfrac{1}{2}E_0\,\hat{\varepsilon}\cdot\vec{d}\right)}_{v^{+}} e^{-i\omega t} + \underbrace{\left(-\tfrac{1}{2}E_0\hat{\varepsilon}^{*}\cdot\vec{d}\right)}_{v} e^{i\omega t}$$

WAVEFUNCTION:

$$|\psi(t)\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle$$

T.D.S.E:

$$i\hbar\,\partial_t|\psi(t)\rangle = (H_0 + H'(t))\,|\psi(t)\rangle$$

E.O.M. FOR C_1, C_2:

$$\begin{cases} i\hbar\, \dot{c}_1 = H'_{12}(t)\, e^{-i\omega_0 t}\, c_2 & (\omega_0 = \omega_2 - \omega_1) \\ i\hbar\, \dot{c}_2 = H'_{21}(t)\, e^{i\omega_0 t}\, c_1 \end{cases}$$

EXPAND $H'(t)$:

$$\begin{cases} i\hbar \dot{c}_1 = \left(v^{+}_{12}\, e^{-i(\omega+\omega_0)t} + v_{12}\, e^{i(\omega-\omega_0)t} \right) c_2 \\ i\hbar \dot{c}_2 = \left(v^{+}_{21}\, e^{-i(\omega-\omega_0)t} + v_{21}\, e^{i(\omega+\omega_0)t} \right) c_1 \end{cases}$$

"ROTATING WAVE APPROXIMATION":

$e^{\pm i(\omega+\omega_0)t}$ OSCILLATES RAPIDLY $\rightarrow$ NEGLIGIBLE EFFECT

$$\begin{cases} i\hbar \dot{c}_1 = v_{12}\, e^{i(\omega-\omega_0)t}\, c_2 \\ i\hbar \dot{c}_2 = v^{+}_{21}\, e^{-i(\omega-\omega_0)t}\, c_1 \end{cases}$$

RABI FREQUENCY Ω:

$$\frac{\hbar\Omega}{2} \equiv v_{12} = -\frac{1}{2} E_0 \langle 1 | \hat{\varepsilon}^{*} \cdot \vec{d}\, | 2 \rangle$$

$$\frac{\hbar\Omega^{*}}{2} = v^{+}_{21} = -\frac{1}{2} E_0 \langle 2 | \hat{\varepsilon} \cdot \vec{d}\, | 1 \rangle$$

DETUNING $\delta = \omega - \omega_0$

$$\begin{cases} i\dot{c}_1 = \frac{1}{2}\Omega\, e^{i\delta t}\, c_2 \\ i\dot{c}_2 = \frac{1}{2}\Omega^{*} e^{-i\delta t}\, c_1 \end{cases}$$

SOLUTION

$$\ddot{c}_2 = \frac{d}{dt}\dot{c}_2 = \frac{d}{dt}\left(-i\frac{\Omega^*}{2}e^{-i\delta t}c_1\right)$$

$$= -\frac{i\Omega^*}{2}\left(-i\delta e^{-i\delta t}c_1 + \underline{\underline{\dot{c}_1}}\, e^{-i\delta t}\right)$$

$$= \underbrace{-\delta\frac{\Omega^*}{2}e^{-i\delta t}c_1}_{i\dot{c}_2} - \frac{\Omega^*}{2}e^{-i\delta t}\frac{\Omega}{2}e^{i\delta t}c_2$$

$$= -i\delta\dot{c}_2 - \left|\frac{\Omega}{2}\right|^2 c_2$$

$$\boxed{\ddot{c}_2 + i\delta\dot{c}_2 + \left|\frac{\Omega}{2}\right|^2 c_2 = 0}$$

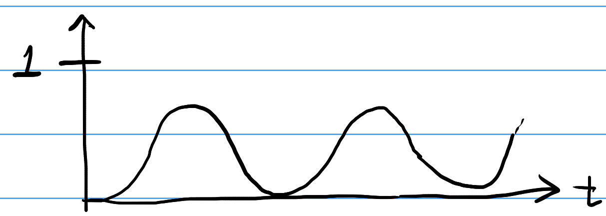
- 2nd ORDER LINEAR ODE W/ CONST COEFFS
- CONSIDER INITIAL CONDITION $c_2(0) = 0$:

  $\rightarrow c_2(t) = A\, e^{-i\delta t/2}\sin[Wt/2]$, $\quad W = \sqrt{\delta^2 + |\Omega|^2}$

- FIND $c_1 = \frac{2i}{\Omega^*}e^{i\delta t}\dot{c}_2$
- GET A USING $c_1(0) = 1$ $\quad \rightarrow A = \frac{-i\Omega^*}{W}$
- IN GENERAL, $|c_1(0)|^2 + |c_2(0)|^2 = 1$

RABI OSCILLATION

$$|c_2(t)|^2 = \frac{|\Omega|^2}{|\Omega|^2+\delta^2}\sin^2\left(\frac{1}{2}\sqrt{\delta^2+|\Omega|^2}\,t\right)$$



RABI OSCILLATION

- OBSERVABLE WHEN SPONT. EMISSION IS NEGLIGIBLE
  - ie. MICROWAVE TRANSITIONS
- BUILDING BLOCK OF ATOM CLOCKS
  & QUANTUM COMPUTERS

PHY 446 SPRING 2022

- HW4 DUE WED, APR 13
- TODAY: ROTATING FRAME TRANSFORMATION

WARM-UP: $\ddot{y} + 2i\dot{y} + y = 0$

$y = e^{\lambda t}$: $\lambda^2 + i\lambda + 1 = 0$

$$\lambda = \frac{-2i \pm \sqrt{-4-4}}{2} = -i \pm i\sqrt{2}$$

$$y = e^{-it} e^{\pm i\sqrt{2}t} \rightarrow e^{-it}\left(A\cos(t\sqrt{2}) + B\sin(t\sqrt{2})\right)$$

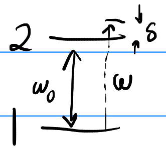
$y(0) = 0$, $\dot{y}(0) = 1$

$\hookrightarrow A = 0$ $\quad\hookrightarrow$ $1 = \frac{d}{dt}\left[e^{-it} B\sin(t\sqrt{2})\right]$

$$= \left[-i\sin(t\sqrt{2}) + \sqrt{2}\cos(t\sqrt{2})\right] B e^{-it}$$

$$= \sqrt{2}B \rightarrow \boxed{B = \frac{1}{\sqrt{2}}}$$

TWO-LEVEL ATOM



$\delta = \omega - \omega_0$

$$H(t) = H_0 + H'(t)$$

$$H_0 |n\rangle = \hbar \omega_n |n\rangle \quad , \quad n = 1, 2$$

APPLIED FIELD:

$$H' = v^{\dagger} e^{-i\omega t} + v e^{i\omega t}$$

QUANTUM STATE: $|\psi(t)\rangle = c_1(t) e^{-i\omega_1 t} |1\rangle + c_2(t) e^{-i\omega_2 t} |2\rangle$

T.D.S.E + RWA:

$$i \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\Omega}{2} e^{i\delta t} \\ \frac{\Omega^*}{2} e^{-i\delta t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

RABI OSCILLATION

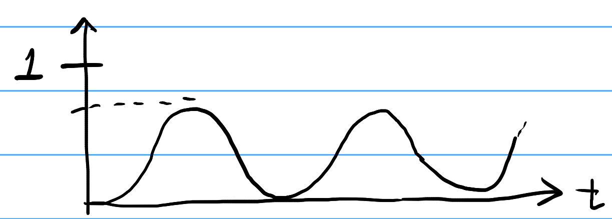
- REDUCE TO 2nd-ORDER EQN FOR c_2:

$$\ddot{c}_2 + i\delta \dot{c}_2 + \frac{1}{2} |\Omega|^2 c_2 = 0$$

SOLUTION FOR $c_2(0) = 0$, $c_1(0) = 1$ $[\Rightarrow \dot{c}_2(0) = -i\Omega^*/2]$

$$c_2(t) = \frac{-i\Omega^*}{\sqrt{\delta^2 + |\Omega|^2}} e^{-i\delta t/2} \sin\left(\frac{1}{2}\sqrt{\delta^2 + |\Omega|^2}\, t\right)$$

PROBABILITY: $|c_2(t)|^2 = \dfrac{|\Omega|^2}{\delta^2 + |\Omega|^2} \sin^2\left(\frac{1}{2}\sqrt{\delta^2 + |\Omega|^2}\, t\right)$



MAX: $\dfrac{|\Omega|^2}{\delta^2 + |\Omega|^2}$

ROTATING FRAME TRANSFORMATION

$$\tilde{c}_1 = c_1 e^{-i\delta t/2}$$
$$\tilde{c}_2 = c_2 e^{i\delta t/2}$$

APPLY TO RWA E.O.M:

$$\begin{cases} i\dot{\tilde{c}}_1 = \frac{1}{2}\delta\,\tilde{c}_1 + \frac{1}{2}\Omega\,\tilde{c}_2 \\ i\dot{\tilde{c}}_2 = \frac{1}{2}\Omega^* c_1 - \delta\,\tilde{c}_2 \end{cases}$$

MATRIX FORM

$$i\hbar\frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \underbrace{\frac{\hbar}{2}\begin{pmatrix}\delta & \Omega\\ \Omega^* & -\delta\end{pmatrix}}_{H_{eff}}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$$

COMPLEX $\Omega = \Omega_1 + i\Omega_2$, $\Omega_{1,2} \in \mathbb{R}$

$$H_{eff} = \frac{\hbar}{2}\begin{pmatrix}\delta & \Omega\\ \Omega^* & -\delta\end{pmatrix} = \frac{\hbar}{2}\left(\Omega_1\sigma_x - \Omega_2\sigma_y + \delta\,\sigma_z\right)$$

$$= \frac{\hbar}{2}(\Omega_1, -\Omega_2, \delta)\cdot\vec{\sigma} = \frac{\hbar}{2}\vec{W}\cdot\vec{\sigma}$$

$$\equiv \vec{W}\cdot\vec{S}$$

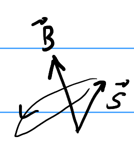
WHERE $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ "PSEUDO-SPIN $\frac{1}{2}$"

COMPARE TO ELECTRON SPIN IN B-FIELD

$$H = -\vec{\mu}\cdot\vec{B} = (2\mu_B\vec{S}/\hbar)\cdot\vec{B}$$

$$= (2\mu_B/\hbar\,\vec{B})\cdot\vec{S} = (\mu_B\vec{B})\cdot\vec{\sigma}$$

SAME DYNAMICS:



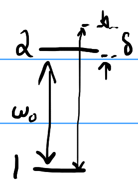
$\langle\vec{S}\rangle$ PRECESSES ABOUT $\vec{B}$

PHY 446 SPRING 2022 LECTURE 21

- HW4 DUE TODAY
- DRAFT OF PAPER DUE MON, APR 18

TODAY: BLOCH SPHERE (FOOT 7.3.2)

WARM-UP: TWO-LEVEL ATOM



$$|\psi(t)\rangle = c_1(t)\, e^{-i\omega_1 t}\, |1\rangle + c_2(t)\, e^{-i\omega_2 t}\, |2\rangle$$

INITIALLY $c_1(0)=1,\ c_2(0)=0$

$$\rightarrow |c_2(t)|^2 = \frac{|\Omega|^2}{\delta^2+|\Omega|^2} \sin^2\left[\frac{1}{2}\sqrt{\delta^2+|\Omega|^2}\ t\right]$$

$|\Omega| =$ RABI FREQ $\propto E_0$ (E1)

- APPLY RESONANT PULSE FOR TIME $T > 0$
- FIND MIN. T S.T. $|C_2(T)|^2 = 1$

$\delta = 0$

$$1 = P_2(T) = \sin^2\left[\tfrac{1}{2}|\Omega| T\right] \Rightarrow \tfrac{1}{2}|\Omega| T = \pi/2$$

$$\Rightarrow \boxed{|\Omega|\, T = \pi} \qquad \text{"}\pi \text{ PULSE"}$$

OVERVIEW:

- TWO-LEVEL MODEL W/O SPONT. EMISS.

  IMPORTANCE:

  1) DIRECT APPLICATIONS:

     RF, MICROWAVE SPECTROSCOPY

     ATOM CLOCKS, QUBITS

  2) CONCEPTUAL:

     FOUNDATION FOR ATOM-LIGHT INTERACTION

     (JUST NEED TO ADD SPONT. EMISS)

· TODAY: "BLOCH SPHERE"

- ANALOGY OF TWO-LEVEL ATOM TO SPIN $\frac{1}{2}$

- SOLVE T.D.S.E. IN PICTURES

REVIEW:

$$H(t) = H_0 + H'(t)$$

$$H'(t) = V^{+} e^{-i\omega t} + V e^{i\omega t}$$

$$\frac{\hbar \Omega}{2} \equiv \langle 1 | V | 2 \rangle$$

RWA:

$$\begin{cases} i \dot{c}_1 = \frac{1}{2} \Omega \, e^{i \delta t} \, c_2 \\ i \dot{c}_2 = \frac{1}{2} \Omega^{*} \, e^{-i \delta t} \, c_1 \end{cases}$$

ROTATING FRAME TRANSFORMATION

· CAN ASSUME $\Omega \in \mathbb{R}$:

FOR $\Omega = \Omega_0 e^{i\theta}$ $\quad (\Omega_0 = |\Omega|)$

USE: $\begin{cases} a_1 = c_1 e^{-i(\delta t + \theta)/2} \\ a_2 = c_2 e^{i(\delta t + \theta)/2} \end{cases}$

$$\Rightarrow i\hbar \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \underbrace{\frac{\hbar}{2} \begin{pmatrix} \delta & \Omega_0 \\ \Omega_0 & -\delta \end{pmatrix}}_{H_{eff}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

FROM HERE ON, LET "Ω" $= \Omega_0$ (REAL)

$$H_{eff} = \frac{\hbar}{2} \Omega \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} + \frac{\hbar}{2} \delta \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

$$= \frac{\hbar}{2} \left(\Omega \, \sigma_x + \delta \, \sigma_z \right) = \frac{\hbar}{2} \vec{W} \cdot \vec{\sigma} \equiv \vec{W} \cdot \vec{S}$$

$$\vec{W} = (\Omega, 0, \delta)$$

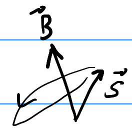
$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ "PSEUDO - SPIN $\frac{1}{2}$"

COMPARE TO ELECTRON SPIN IN B-FIELD

$$H = -\vec{\mu}\cdot\vec{B} \propto \vec{B}\cdot\vec{S}$$

$$\left[\begin{aligned} H &= \left(2\mu_B \vec{S}/\hbar\right)\cdot\vec{B} \\ &= \left(2\mu_B/\hbar\,\vec{B}\right)\cdot\vec{S} = \left(\mu_B\vec{B}\right)\cdot\vec{\sigma} \end{aligned} \right]$$

SAME DYNAMICS:



$\langle\vec{S}\rangle$ PRECESSES ABOUT $\vec{B}$

EFFECTIVE SPIN $\frac{1}{2}$ WAVEFUNCTION

$$|\Psi(t)\rangle = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$$

- T.D.S.E.

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H_{eff}|\Psi(t)\rangle \quad \left[= \frac{\hbar}{2}\vec{W}\cdot\vec{\sigma}\;|\Psi(t)\rangle\right]$$

$$\left[\Rightarrow \begin{cases} \frac{d}{dt}|\Psi(t)\rangle = \frac{-i}{\hbar}H_{eff}|\Psi(t)\rangle \\ \frac{d}{dt}\langle\Psi(t)| = \frac{i}{\hbar}\langle\Psi(t)|H_{eff} \end{cases} \right]$$

- DEFINE THE BLOCH VECTOR

$$\vec{R} = \langle\Psi(t)|\,\vec{\sigma}\,|\Psi(t)\rangle = \left(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle\right)$$

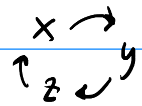
- EQUATION OF MOTION

$$\frac{d}{dt}\langle\vec{\sigma}\rangle \left[\begin{aligned} &= \left(\frac{d}{dt}\langle\Psi|\right)\vec{\sigma}|\Psi(t)\rangle + \langle\Psi|\vec{\sigma}\frac{d}{dt}|\Psi\rangle \\ &= \frac{i}{\hbar}\langle\Psi|H_{eff}\vec{\sigma}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\vec{\sigma}H_{eff}|\Psi\rangle \end{aligned} \right]$$

$$= \frac{i}{\hbar}\langle\Psi|\left[H_{eff}, \vec{\sigma}\right]|\Psi\rangle$$

COMMUTATORS (SPIN):

$$[S_x, S_y] = i\hbar S_z$$



$$[S_y, S_z] = i\hbar S_x \quad \text{etc.}$$

PAULI MATRICES

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \; : \; \left(\frac{\hbar}{2}\right)^2 [\sigma_x, \sigma_y] = i\frac{\hbar^2}{2}\sigma_z$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z \qquad \text{etc.}$$

BACK TO BLOCH VECTOR

$$[H_{eff}, \vec{\sigma}] = [\tfrac{\hbar}{2}\vec{W}\cdot\vec{\sigma}, \vec{\sigma}]$$

$$= -i\hbar\, \vec{W}\times\vec{\sigma}$$

EQN OF MOTION:

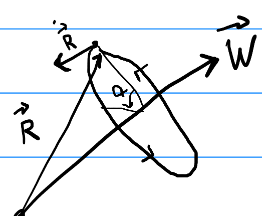
$$\frac{d}{dt}\langle\vec{\sigma}\rangle = \frac{i}{\hbar}\langle\Psi|[H_{eff}, \vec{\sigma}]|\Psi\rangle$$

$$= \langle\Psi|\vec{W}\times\vec{\sigma}|\Psi\rangle = \vec{W}\times\langle\vec{\sigma}\rangle$$

i.e,

$$\frac{d\vec{R}}{dt} = \vec{W}\times\vec{R} \qquad , \qquad \vec{W} = (\Omega, 0, \delta)$$

PRECESSION



- $\dot{\vec{R}}$ is $\perp$ to $\vec{R}$, $\alpha = |\vec{W}|t = \sqrt{\Omega^2+\delta^2}\,t$

COMPONENTS OF THE BLOCH VECTOR

$$R_x = \langle \Psi | \sigma_x | \Psi \rangle = (a_1^*, a_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^*, a_2^*) \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$$

$$= a_1 a_2^* + a_2 a_1^*$$

$$R_y = \langle \Psi | \sigma_y | \Psi \rangle = (a_1^*, a_2^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^*, a_2^*) \begin{pmatrix} -ia_2 \\ ia_1 \end{pmatrix}$$

$$= i(a_1 a_2^* - a_2 a_1^*)$$

$$R_z = \langle \Psi | \sigma_z | \Psi \rangle = (a_1^*, a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^*, a_2^*) \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix}$$

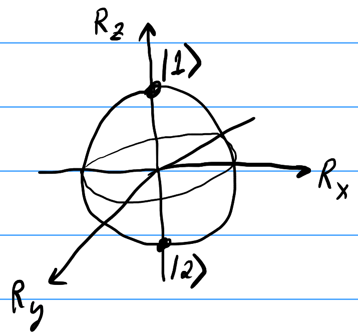
$$= |a_1|^2 - |a_2|^2$$

NORMALIZATION: $|a_1|^2 + |a_2|^2 = 1 \Rightarrow |\vec{R}| = 1$ (HWS)

EXAMPLE: $a_1 = 1,\ a_2 = 0 \Rightarrow R_z = 1$

$a_1 = 0,\ a_2 = 1 \Rightarrow R_z = -1$

BLOCH SPHERE: $\vec{R}$ ON THE UNIT SPHERE



NORTH POLE: $|1\rangle$

SOUTH POLE: $|2\rangle$

XY PLANE: EQUAL SUPERPOSITIONS

$$\frac{1}{\sqrt{2}}|1\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|2\rangle$$

EXAMPLE

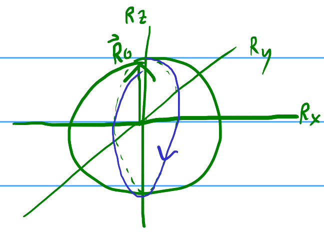
GIVEN: ATOM IN $|1\rangle$ AT $t=0$

RESONANT LIGHT $(\delta=0)$, RABI FREQ. Ω

a) DRAW $\vec{R}(0)$, $\vec{W}$ ON BLOCH SPHERE

b) DRAW $\vec{R}(t)$ FOR $t>0$

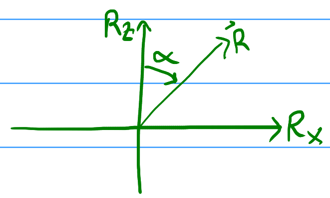
c) FIND SMALLEST $T>0$ S.T. ATOM IN $|2\rangle$



a,b) $t=0$: $\vec{R}(0)=(0,0,1)$

$\vec{W}=(\Omega,0,0)$

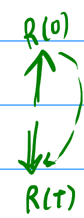
c)



$\alpha = \Omega t$

STATE $|2\rangle \Rightarrow \alpha = \boxed{\pi = \Omega T}$

"π PULSE"



EXCITED STATE PROBABILITY

- NORMALIZATION: $|a_1|^2 + |a_2|^2 = 1 \rightarrow |a_1|^2 = 1 - |a_2|^2$

$$R_z = 1 - 2|a_2|^2 \Rightarrow |a_2|^2 = \tfrac{1}{2}(1 - R_z)$$

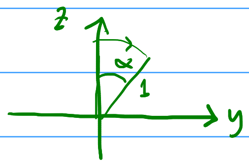
$$\Rightarrow |a_1|^2 = \tfrac{1}{2}(1 + R_z)$$

π PULSE EXAMPLE, CONTINUED $(\delta = 0)$

c) FIND $R_z(t)$

HINT: RE-DRAW $\vec{R}(t)$ IN yz PLANE

d) FIND $|a_2(t)|^2$

c) 

$$R_z(t) = \cos(\alpha(t)) = \cos(\Omega t)$$

d) $$|a_2(t)|^2 = \tfrac{1}{2}(1 - R_z(t)) = \tfrac{1}{2}\left[1 - \cos(\Omega t)\right]$$

$$= \sin^2\left(\tfrac{1}{2}\Omega t\right)$$

$$\left(\begin{aligned} &\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \\ &\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \\ &\sin^2\left(\tfrac{1}{2}x\right) = \tfrac{1}{2}(1 - \cos x) \end{aligned}\right)$$

PHY 446 SPRING 2022 LECTURE 22

TODAY: OPTICAL BLOCH EQUATIONS (FOOT 7.5.2)

- PAPER DRAFT DUE
- HW5 NEXT MON, APR 25
- FINAL PAPER DUE WED, APR 27
- MAY 2,4 – FINAL PRESENTATIONS (15 min)

WARM-UP: TWO-LEVEL MODEL

a) GIVEN BLOCH VECTOR $\vec{R} = (R_x, R_y, R_z)$

FIND PROB. of $|2\rangle$, i.e. $|a_2|^2$

b) FOR $|\Psi(0)\rangle = |1\rangle$, RESONANT PULSE $(\delta = 0)$

FIND $\vec{R}(t)$ AND $P_2(t)$

a) $|a_1|^2 + |a_2|^2 = 1 \rightarrow |a_1|^2 = 1 - |a_2|^2$

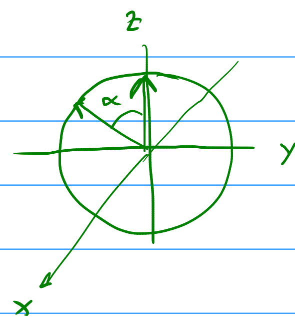
$R_z = \underline{|a_1|^2 - |a_2|^2} = 1 - 2|a_2|^2$

$$\boxed{|a_2|^2 = \tfrac{1}{2}(1 - R_z)}$$

b) $a_1(0) = 1, \quad a_2(0) = 0$

$\vec{R}(0) = (0, 0, 1)$

$\vec{W} = (\Omega, 0, \delta) = (\Omega, 0, 0)$



$\vec{R}(t) = -\sin\alpha\, \hat{e}_y + \cos\alpha\, \hat{e}_z, \quad \alpha = |\vec{W}|t = \Omega t$

$P_2(t) = \tfrac{1}{2}(1 - R_z(t)) = \tfrac{1}{2}[1 - \cos(\Omega t)] = \boxed{\sin^2(\tfrac{1}{2}\Omega t)}$ ✓ RABI OSCILLATION

NOTE: $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$

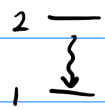
$\sin^2(\tfrac{1}{2}x) = \tfrac{1}{2}(1 - \cos x)$

ROTATING WAVE APPROX:

$$|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$i\hbar \frac{d}{dt}|\Psi\rangle = H|\Psi\rangle = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}|\Psi\rangle$$

SPONTANEOUS EMISSION



ATOM RANDOMLY DECAYS FROM $|2\rangle \to |1\rangle$

- RATE Γ
- EMITS A PHOTON

FREE DECAY ($\delta = 0$, $\Omega = 0$):

$P_2(t)$ = EXCITED STATE PROB.

$\Gamma \delta t \approx P(\text{DECAY DURING } \delta t \mid \text{STARTED IN } |2\rangle) \ll 1$

$$P_2(t+\delta t) = (1 - \Gamma \delta t)\, P_2(t) = P_2(t) - \Gamma \delta t\, P_2(t)$$

$$\Rightarrow \delta P_2 = P_2(t+\delta t) - P_2(t) = -\Gamma \delta t\, P_2(t)$$

$$\frac{dP_2}{dt} = -\Gamma P_2$$

SOLUTION: $P_2(t) = P_2(0)\, e^{-\Gamma t}$

MODIFIED DYNAMICS

1) $|a_2(t)|^2 \sim e^{-\Gamma t} \Rightarrow a_2(t) \sim e^{-\Gamma t/2} = e^{-i(-i\Gamma/2)t}$

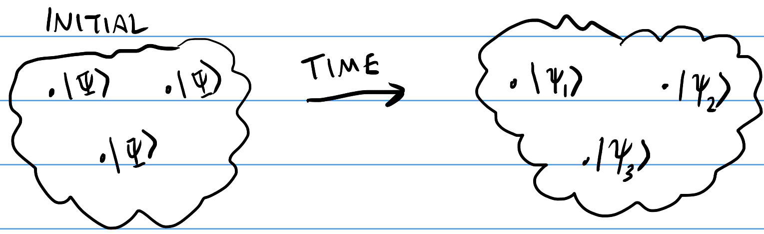
$$H \to \tilde{H} = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta - i\Gamma \end{pmatrix}$$ – NON-HERMITIAN

☆ 2) QUANTUM JUMPS: SPONTANEOUS CHANGE FROM $|2\rangle \to |1\rangle$

- NOT INCLUDED IN T.D.S.E.
- $|\Psi\rangle$ NO LONGER DETERMINISTIC

ENSEMBLE OF ATOMS

- CONSIDER N ATOMS INITIALLY IN STATE $|\Psi\rangle = a_1|1\rangle + a_2|2\rangle$
- TIME EVOLVE: STATE VECTORS DIFFER DUE TO SPONT. DECAY



OBSERVABLES

- LET D = HERMITIAN OPERATOR $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

IN STATE $|\Psi\rangle$:

$$\langle D\rangle = (a_1^*, a_2^*)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^*, a_2^*)\begin{pmatrix} a_1 a + a_2 b \\ a_1 c + a_2 d \end{pmatrix}$$

$$= a\,|a_1|^2 + b\,a_2 a_1^* + c\,a_1 a_2^* + d\,|a_2|^2$$

ENSEMBLE AVG:

$$\overline{\langle D\rangle} = \frac{1}{N}\sum_{\alpha=1}^{N}\langle D_\alpha\rangle = a\,\overline{a_1 a_1^*} + b\,\overline{a_2 a_1^*} + c\,\overline{a_1 a_2^*} + d\,\overline{a_2 a_2^*}$$

- DEPENDS ON $\overline{a_i a_j^*} = \langle a_i a_j^*\rangle_{ens} \neq \langle a_i\rangle_{ens}\langle a_j^*\rangle_{ens}$

- DETERMINES ENSEMBLE PROPERTIES
  - ABSORPTION, INDEX of REFRACTION
  - AVG. OPTICAL FORCES ON ATOMS

DENSITY MATRIX

- CONSIDER TWO-LEVEL SYSTEM

$$|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- DEFINE

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (a_1^*, a_2^*)$$

$$= \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_2 a_1^* & |a_2|^2 \end{pmatrix} \equiv \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

- RELATED TO BLOCH VECTOR $\vec{R} = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$

$$= \left(a_1 a_2^* + a_2 a_1^*, \; i(a_1 a_2^* - a_2 a_1^*), \; |a_1|^2 - |a_2|^2 \right)$$

- ρ IS MORE GENERAL THAN $\vec{R}$ (BEYOND 2-LEVEL)

OBSERVABLES

$$\langle D\rangle = \langle\Psi|D|\Psi\rangle = \sum_{mn} \langle\Psi|m\rangle\langle m|D|n\rangle\langle n|\Psi\rangle = \sum_{mn} \underbrace{\langle n|\Psi\rangle\langle\Psi|m\rangle}_{\rho_{nm}} D_{mn}$$

$$= \sum_n (\rho D)_{nn} = tr(\rho D) = tr(D\rho)$$

- general: $tr(AB) = tr(BA)$

ENSEMBLE AVG:

$$\overline{\langle D\rangle} = \frac{1}{N}\sum_{\alpha=1}^{N} \langle\psi_\alpha|D|\psi_\alpha\rangle = \frac{1}{N}\sum_{\alpha=1}^{N} tr(\rho_\alpha D) = tr(\bar{\rho} D)$$

$$\bar{\rho} = \frac{1}{N}\sum_{\alpha=1}^{N} \rho_\alpha \quad \text{ENSEMBLE AVG DENSITY MATRIX}$$

- $\bar{\rho}$ DETERMINES ENSEMBLE AVG. VALUES

PURE VS. MIXED STATES

- PURE: $\rho = |\Psi\rangle\langle\Psi| \Rightarrow \rho^2 = \rho$
- MIXED: $\rho \neq |\Psi\rangle\langle\Psi|$ FOR ANY $|\Psi\rangle \Rightarrow \rho^2 \neq \rho$

EQN OF MOTION FOR DENSITY MATRIX

FOR PURE STATE $\rho = |\Psi\rangle\langle\Psi|$,

$$i\hbar \frac{d}{dt}\rho = \left(i\hbar \frac{d}{dt}|\Psi\rangle\right)\langle\Psi| + |\Psi\rangle\, i\hbar \frac{d}{dt}\langle\Psi| = H|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|H$$
$$= [H, \rho]$$

WITH SPONTANEOUS EMISSION:

- LET ρ = ensemble avg = $\bar{\rho}$

$$\left.\frac{d\rho}{dt}\right|_{\text{spont. emiss.}} = \begin{pmatrix} \Gamma\rho_{22} & -\frac{\Gamma}{2}\rho_{12} \\ -\frac{\Gamma}{2}\rho_{21} & -\Gamma\rho_{22} \end{pmatrix} \equiv L(\rho)$$

"MASTER EQUATION":

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}[H, \rho] + L(\rho)$$

EXERCISE: CALCULATE $\frac{1}{\hbar}[H, \rho]$

$$H = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}, \quad \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{\hbar}H\rho = \frac{1}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \delta a + \Omega c & \delta b + \Omega d \\ \Omega a - \delta c & \Omega b - \delta d \end{pmatrix}$$

$$\frac{1}{\hbar}\rho H = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \delta a + \Omega b & \Omega a - \delta b \\ \delta c + \Omega d & \Omega c - \delta d \end{pmatrix}$$

$$\frac{1}{\hbar}[H, \rho] = \frac{1}{2}\begin{pmatrix} \Omega(c-b) & \Omega(d-a) + 2\delta b \\ \Omega(a-d) - 2\delta c & \Omega(b-c) \end{pmatrix}$$

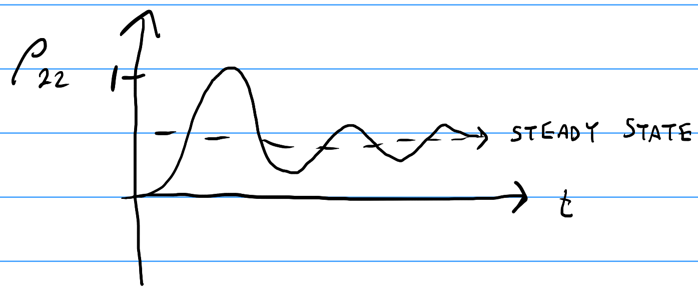
$$\frac{d\rho}{dt} = \begin{pmatrix} -i\frac{\Omega}{2}(\rho_{21} - \rho_{12}) & -i\frac{\Omega}{2}(\rho_{22} - \rho_{11}) - i\delta\rho_{12} \\ -i\frac{\Omega}{2}(\rho_{11} - \rho_{22}) + i\delta\rho_{21} & -i\frac{\Omega}{2}(\rho_{12} - \rho_{21}) \end{pmatrix}$$

DAMPED RABI OSCILLATIONS

- CONSIDER $\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, ALL ATOMS IN $|1\rangle$

- $\delta = 0$, $\underline{\Gamma \ll \Omega}$

$$\Rightarrow \quad \rho_{22}(t) \approx \frac{1}{2} \frac{\Omega^2}{\Omega^2 + \Gamma^2/2} \left[1 - e^{-\frac{3}{4}\Gamma t} \cos(\Omega t) \right]$$



PHY 446 SPRING 2022 LECTURE 23

TODAY: MASTER EQN & OPTICAL BLOCH EQNS

WARM-UP: 2-LEVEL ATOM

50% CHANCE OF $|\Psi\rangle = |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

50% CHANCE OF $|\Psi\rangle = |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

FIND

1. a) DENSITY MATRICES ρ_1, ρ_2
   b) AVG DENSITY MATRIX ρ
   c) $tr(\rho_1)$, $tr(\rho_2)$, $tr(\rho)$

a) $\rho_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1,1) = \boxed{\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}}$

$\rho_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}(1,-i) = \boxed{\frac{1}{2}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix}}$

b)

$\rho = 0.5\rho_1 + 0.5\rho_2 = \boxed{\begin{pmatrix}1/2 & \frac{1}{4}(1-i)\\ \frac{1}{4}(1+i) & 1/2\end{pmatrix}}$

c) $tr(\rho_1) = 1$, $tr(\rho_2) = 1$, $tr(\rho) = 1$

2. a) BLOCH VECTORS $\vec{R}_1$, $\vec{R}_2$
   b) AVG BLOCH VECTOR $\vec{R}$
   c) $|\vec{R}_1|$, $|\vec{R}_2|$, $|\vec{R}|$

a)
$$\left.\begin{aligned} R_{1x} &= (\rho_1)_{12} + (\rho_1)_{21} = 1 \\ R_{1y} &= i[(\rho_1)_{12} - (\rho_1)_{21}] = 0 \\ R_{1z} &= (\rho_1)_{11} - (\rho_1)_{22} = 0 \end{aligned}\right\} \vec{R}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

2 a) CONT:

$$\left.\begin{aligned} R_{2x} &= (\rho_2)_{12} + (\rho_2)_{21} = 0 \\ R_{2y} &= i[(\rho_2)_{12} - (\rho_2)_{21}] = 1 \\ R_{2z} &= (\rho_2)_{11} - (\rho_2)_{22} = 0 \end{aligned}\right\} \vec{R}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

b) $\vec{R} = 0.5\vec{R}_1 + 0.5\vec{R}_2 = \boxed{\begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}}$

c) $|R_1| = 1$, $|R_2| = 1$, $|R| = \sqrt{0.25 \times 2} \approx \boxed{\frac{1}{\sqrt{2}}} = 0.71$

NOTE:

PURE	MIXED
$\lvert\vec{R}\rvert = 1$	$\lvert\vec{R}\rvert < 1$
$\rho^2 = \rho$	$\rho^2 \neq \rho$

ENSEMBLE AVGS

- TWO WAYS TO INTERPRET ρ:

1. N ATOMS, W/ KNOWN STATES $|\psi_i\rangle$:

$$\rho = \sum_{i=1}^{N} \frac{1}{N} |\psi_i\rangle\langle\psi_i|$$

2. ONE ATOM, POSSIBLE STATES $|\psi_i\rangle$, PROB. P_i:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

[3. PARTIAL TRACE $\rho = tr_R \left(|\psi_{AR}\rangle\langle\psi_{AR}| \right)$]

EQN OF MOTION FOR DENSITY MATRIX

T.D.S.E:

$$i\hbar \frac{d}{dt}|\Psi\rangle = H|\Psi\rangle \rightarrow -i\hbar \frac{d}{dt}\langle\Psi| = \langle\Psi|H$$

FOR PURE STATE $\rho = |\Psi\rangle\langle\Psi|$,

$$i\hbar \frac{d}{dt}\rho = \left(i\hbar \frac{d}{dt}|\Psi\rangle\right)\langle\Psi| + |\Psi\rangle\, i\hbar \frac{d}{dt}\langle\Psi| = H|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|H$$

$$= [H, \rho]$$

WITH SPONTANEOUS EMISSION:

- LET ρ = ensemble avg
- RECAL: 1) $a_2 \sim e^{-\Gamma t/2} \Rightarrow \rho_{22} \sim e^{-\Gamma t}$

$$\Rightarrow \rho_{12} = \overline{a_1 a_2^*} \sim e^{-\Gamma t/2}, \quad \rho_{21} \sim e^{-\Gamma t/2}$$

2) QUANTUM JUMPS $|2\rangle \rightarrow |1\rangle$ AT RATE Γ

→ STOCHASTIC TRAJECTORIES

AVERAGE*:

$$\left.\frac{d\rho}{dt}\right|_{\text{spont. emiss.}} = \underbrace{\begin{pmatrix} 0 & -\frac{\Gamma}{2}\rho_{12} \\ -\frac{\Gamma}{2}\rho_{21} & -\Gamma\rho_{22} \end{pmatrix}}_{\text{DECAY}} + \underbrace{\begin{pmatrix} \Gamma\rho_{22} & 0 \\ 0 & 0 \end{pmatrix}}_{\text{QUANTUM JUMPS}}$$

$$\equiv L(\rho)$$

* MØLMER et al 1993

"MASTER EQUATION":

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}[H, \rho] + L(\rho)$$

CALCULATE $\frac{1}{\hbar}[H,\rho]$

$$H = \frac{\hbar}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}, \quad \rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{\hbar}H\rho = \frac{1}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \delta a + \Omega c & \delta b + \Omega d \\ \Omega a - \delta c & \Omega b - \delta d \end{pmatrix}$$

$$\frac{1}{\hbar}\rho H = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \delta a + \Omega b & \Omega a - \delta b \\ \delta c + \Omega d & \Omega c - \delta d \end{pmatrix}$$

$$\frac{1}{\hbar}[H,\rho] = \frac{1}{2}\begin{pmatrix} \Omega(c-b) & \Omega(d-a) + 2\delta b \\ \Omega(a-d) - 2\delta c & \Omega(b-c) \end{pmatrix}$$

$$\frac{d\rho}{dt} = \begin{pmatrix} -i\frac{\Omega}{2}(\rho_{21}-\rho_{12}) & -i\frac{\Omega}{2}(\rho_{22}-\rho_{11}) - i\delta\rho_{12} \\ -i\frac{\Omega}{2}(\rho_{11}-\rho_{22}) + i\delta\rho_{21} & -i\frac{\Omega}{2}(\rho_{12}-\rho_{21}) \end{pmatrix} + L(\rho)$$

BLOCH VECTOR

- LET $\vec{R} = (u,v,w)$ [FOOT : $v \to -v$]

$$\left.\frac{d\vec{R}}{dt}\right|_{\text{spont. emiss}} = \frac{d}{dt}\begin{pmatrix} \rho_{12}+\rho_{21} \\ i(\rho_{12}-\rho_{21}) \\ \rho_{11}-\rho_{22} \end{pmatrix}_{\text{spont. emiss}} = \begin{pmatrix} -\frac{\Gamma}{2}(\rho_{12}+\rho_{21}) \\ -\frac{\Gamma}{2}\, i(\rho_{12}-\rho_{21}) \\ \Gamma\rho_{22} + \Gamma\rho_{22} \end{pmatrix}$$

$\rho_{22} = \frac{1-w}{2}$

$$= \begin{pmatrix} -\frac{\Gamma}{2}u \\ -\frac{\Gamma}{2}v \\ \Gamma(1-w) \end{pmatrix}$$

OPTICAL BLOCH EQNS

$$\frac{d\vec{R}}{dt} = \vec{W}\times\vec{R} - \Gamma\begin{pmatrix} u/2 \\ v/2 \\ w-1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \dot{u} = \delta v - \frac{\Gamma}{2}u \\ \dot{v} = \delta u - \Omega w - \frac{\Gamma}{2}v \\ \dot{w} = \Omega v - \Gamma(w-1) \end{cases}$$

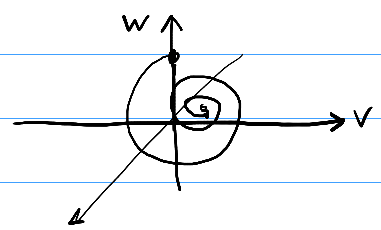
DAMPED RABI OSCILLATIONS

Ex. $\delta = 0$, $\Gamma \ll \Omega$, $\vec{R}(0) = (0,0,1)$

$$\dot{u} = -\frac{\Gamma}{2}u \rightarrow u(t) = 0$$

$$\begin{cases} \dot{v} = -\Omega w - \frac{\Gamma}{2}w \\ \dot{w} = \Omega v - \Gamma(w-1) \end{cases}$$

→ DAMPED CIRCULAR MOTION IN V-W PLANE

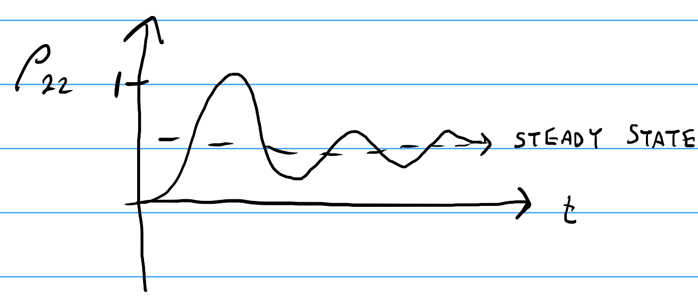


$$w_{ss} = \frac{\Gamma^2/2}{\Omega^2+\Gamma^2/2}$$

$$v_{ss} = \frac{\Omega\Gamma}{\Omega^2+\Gamma^2/2}$$

- SPIRALS TOWARD STEADY STATE

$$\rho_{22}(t) = \frac{1-w}{2} \approx \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Gamma^2/2}\left[1 - e^{-\frac{3}{4}\Gamma t}\cos(\Omega t)\right]$$



TODAY: STEADY-STATE SOLUTION (FOOT 7.5.2)

ABSORPTION (FOOT 7.6)

OPTICAL BLOCH EQNS

- AVG $\vec{R} = (u, v, w)$

$$\begin{cases} \dot{u} = \delta v - \frac{\Gamma}{2} u \\ \dot{v} = \delta u - \Omega w - \frac{\Gamma}{2} v \\ \dot{w} = \Omega v - \Gamma (w-1) \end{cases}$$

EXERCISE: $|\psi(0)\rangle = |2\rangle$, $\Omega = 0, \delta = 0$

a) $\vec{R}(0)$

b) $\vec{R}(t)$, PLOT,

c) Steady-state $\vec{R}(t \to \infty)$

d) $\rho_{22}(t)$, $\rho_{22}(t \to \infty)$

a) $\vec{R} = (a_1 a_2^* + a_2 a_1^*, \ i(a_1 a_2^* - a_2 a_1^*), |a_1|^2 - |a_2|^2) = (0, 0, -1)$

b) OBE w/ $\Omega = 0, \delta = 0$:

$\dot{u} = -\Gamma u/2 \to u = 0$

$\dot{v} = -\Gamma v/2 \to v = 0$

$\dot{w} = -\Gamma(w-1) \to \dot{w} + \Gamma w = \Gamma$

homogeneous: $w = Ae^{-\Gamma t}$

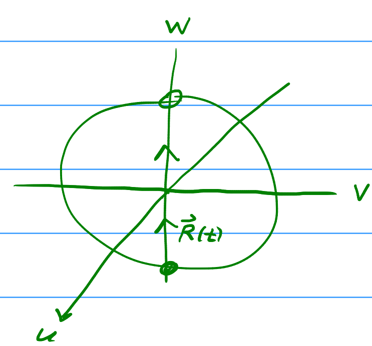
Particular: $w = 1$

general: $w(t) = Ae^{-\Gamma t} + 1$

initial: $-1 = w(0) = A + 1 \Rightarrow A = -2$

$$\boxed{w(t) = 1 - 2e^{-\Gamma t}}$$

$$\left[\begin{aligned} 1 &= \rho_{11} + \rho_{22} \\ \rho_{11} &= 1 - \rho_{22} \\ w &= \rho_{11} - \rho_{22} = 1 - 2\rho_{22} \end{aligned}\right]$$



c) $w_{s\text{-}s} = 1$

$$\left[\begin{aligned} &d) \ \rho_{22}(t) = e^{-\Gamma t} \to 0 \\ &\rho_{11}(t) = 1 - \rho_{22} = 1 - e^{-\Gamma t} \to 1 \end{aligned}\right]$$

STEADY-STATE SOLUTION

- FOR $\delta = 0$

$$\begin{cases} \dot{u} = -\frac{\Gamma}{2} u = 0 \quad \rightarrow u = 0 \\ \dot{v} = -\Omega w - \frac{\Gamma}{2} v = 0 \\ \dot{w} = \Omega v - \Gamma (w-1) = 0 \end{cases}$$

$$\begin{pmatrix} -\Gamma/2 & -\Omega \\ \Omega & -\Gamma \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -\Gamma \end{pmatrix}$$

$$\begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{\Gamma^2/2 + \Omega^2} \begin{pmatrix} -\Gamma & \Omega \\ -\Omega & -\Gamma/2 \end{pmatrix} \begin{pmatrix} 0 \\ -\Gamma \end{pmatrix} = \frac{1}{\frac{\Gamma^2}{2} + \Omega^2} \begin{pmatrix} -\Omega\Gamma \\ \Gamma^2/2 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{s.s.} = \frac{1}{\Omega^2/2 + \Gamma^2/4} \begin{pmatrix} 0 \\ -\Omega\Gamma/2 \\ \Gamma^2/4 \end{pmatrix}$$

- FOR $\delta \neq 0$: 3 COUPLED EQNS,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{s.s.} = \frac{1}{\delta^2 + \Omega^2/2 + \Gamma^2/4} \begin{pmatrix} \Omega\delta \\ -\Omega\Gamma/2 \\ \delta^2 + \Gamma^2/4 \end{pmatrix}$$

EXCITED STATE PROB

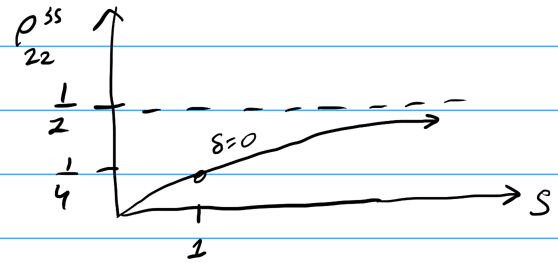
$$\rho_{22}^{(s.s.)} = \frac{1 - w_{ss}}{2} = \frac{1}{2} \frac{\Omega^2/2}{\delta^2 + \Omega^2/2 + \Gamma^2/4}$$

$$= \frac{1}{2} \frac{2\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2 + (2\delta/\Gamma)^2}$$

$$= \frac{1}{2} \frac{S}{1 + S + (2\delta/\Gamma)^2}$$

$S \equiv 2\Omega^2/\Gamma^2$ "SATURATION PARAMETER"

STEADY STATE EXCITED STATE FRACTION



SATURATION INTENSITY

- For E1 or M1 transitions, $\Omega^2 \propto I$ (intensity)

$$\rightarrow s = 2\Omega^2/\Gamma^2 \equiv I/I_{SAT}$$

for some I_{SAT}

WHAT IS I_{SAT}?

- Solve: $I_{SAT} = I/s = \dfrac{\Gamma^2 I}{2\Omega^2}$

- Relate Ω^2 to I (for E1):

$$\vec{E}(0,t) = \mathrm{Re}\left[E_0 \hat{\varepsilon} e^{-i\omega t}\right] = \frac{1}{2} E_0 \left(\hat{\varepsilon} e^{-i\omega t} + \hat{\varepsilon}^* e^{i\omega t} \right)$$

$$\hbar \Omega e^{i\phi} = - E_0 \langle 1 | \hat{\varepsilon}^* \cdot \vec{d} \, | 2 \rangle$$

$$\Omega^2 = \frac{E_0^2 \, |\langle 1 | \hat{\varepsilon}^* \cdot \vec{d} | 2 \rangle|^2}{\hbar^2} = \frac{E_0^2 |D_{12}|^2}{\hbar^2}$$

$$D_{12} = \langle 1 | \hat{\varepsilon} \cdot \vec{d} \, | 2 \rangle$$

Intensity: $I = c\,\epsilon_0 \langle \vec{E}^2 \rangle_t = c\,\epsilon_0 \left\langle \frac{1}{4} E_0^2 \left(\hat{\varepsilon}\cdot\hat{\varepsilon}\, e^{-i2\omega t} + 2 + \hat{\varepsilon}^*\cdot\hat{\varepsilon}^* e^{2i\omega t} \right) \right\rangle_t$

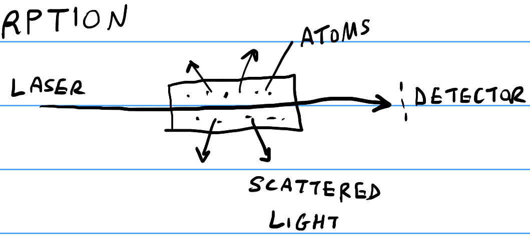
$$= \frac{1}{2} c\,\epsilon_0 E_0^2$$

$$\rightarrow E_0^2 = \frac{2I}{\epsilon_0 c}$$

$$\Omega^2 = \frac{2I}{\epsilon_0 c} |D_{12}|^2/\hbar^2 \Rightarrow \frac{I}{\Omega^2} = \frac{\hbar^2 \epsilon_0 c}{2\,|D_{12}|^2}$$

$$\boxed{I_{SAT} = \frac{\Gamma^2 \hbar^2 \epsilon_0 c}{4\,|D_{12}|^2}}$$

ABSORPTION



BEER'S LAW:

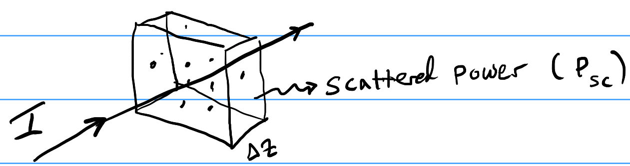
$$\frac{dI}{dz} = -aI$$

a = ABSORPTION COEFFICIENT

(FOOT USES KAPPA κ)

FIND a

- CONSIDER A THIN SLICE



A = AREA

N = NUM. ATOMS = DENSITY × VOL = $n_a \times A \times \Delta z$

\# PHOTONS SCATTERED $\quad R_{sc} = N \Gamma \rho_{22}^{(s.s.)} = N \frac{\Gamma}{2} \frac{I/I_{SAT}}{1+S+(2\delta/\Gamma)^2}$

$$\Delta I = -\frac{P_{sc}}{A} = -\frac{\hbar\omega}{A} R_{sc} = -\frac{\hbar\omega}{A} N \Gamma \rho_{22} = -\hbar\omega(n_a \Delta z)\Gamma\rho_{22}$$

$$\frac{dI}{dz} = \underbrace{-\frac{\hbar\omega\, n_a \Gamma}{2} \frac{1/I_{SAT}}{1+S+(2\delta/\Gamma)^2}}_{a} I \equiv -aI$$

$$a \approx \underbrace{\frac{\hbar\omega_0 \Gamma n_a}{2 I_{SAT}}}_{a_0} \frac{1}{1+S+(2\delta/\Gamma)^2} = \frac{a_0}{1+S+(2\delta/\Gamma)^2}$$

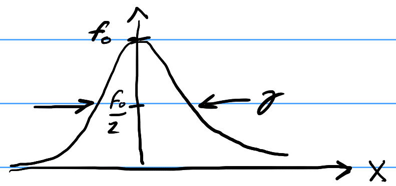
$$= \boxed{\frac{a_0}{1+S} \frac{1}{1+\left[\frac{2\delta}{\Gamma\sqrt{1+S}}\right]^2}}$$

LORENTZIAN

• GENERAL: $f(x) = \dfrac{f_0}{1+(2x/\gamma)^2}$

f_0 = MAXIMUM

γ = FWHM



ABSORPTION:

$$a(\delta) = \frac{a_0/(1+s)}{1+\left[\frac{2\delta}{\Gamma\sqrt{1+s}}\right]^2}$$

MAX: $a_0/(1+s)$

• DECREASES WITH S → "SATURATION"

FWHM: $\Gamma\sqrt{1+s}$

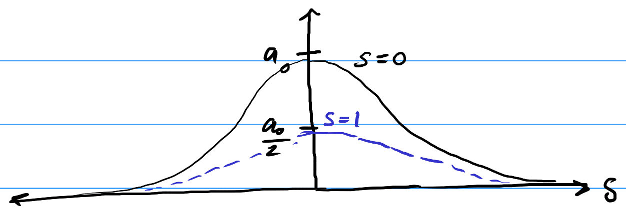
• INCREASES WITH S → "POWER BROADENING"

EX. FOR $I = I_{SAT}$, FIND: $(s=1)$

a) MAX a $= a_0/(1+s) = a_0/2$

b) FWHM $= \Gamma\sqrt{1+s} = \Gamma\sqrt{2}$

PLOT: ABSORPTION VS. δ

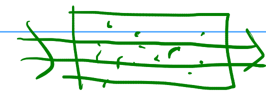


PHY 446 LECTURE 25

TODAY

- DOPPLER BROADENING
- OPTICAL FORCES

WARM-UP: $\frac{dI}{dz} = -aI$ (BEER'S LAW)



GIVEN $I(z=0) = I_0$

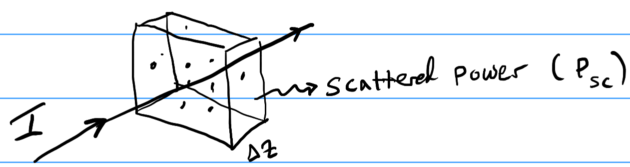
IF a = CONST (INDEP. OF I)

FIND: $I(z)$

$$I(z) = I_0 e^{-az}$$

ABSORPTION

- CONSIDER A THIN SLICE OF "FROZEN GAS"



A = AREA

N = NUM. ATOMS = DENSITY × VOL = $n_a \times A \times \Delta z$

PHOTONS SCATTERED $R_{sc} = N \Gamma \rho_{22}^{(s.s.)} = N \frac{\Gamma}{2} \frac{I/I_{SAT}}{1+S+(2\delta/\Gamma)^2}$

$$\Delta I = -\frac{P_{sc}}{A} = -\frac{\hbar\omega}{A} R_{sc} = -\frac{\hbar\omega}{A} N \Gamma \rho_{22} = -\hbar\omega (n_a \Delta z)\Gamma \rho_{22}$$

$$\frac{dI}{dz} = -\hbar\omega n_a \Gamma \rho_{22}$$

$$= \underbrace{-\frac{\hbar\omega n_a \Gamma}{2} \frac{1/I_{SAT}}{1+S+(2\delta/\Gamma)^2}}_{a} I \equiv -aI$$

$$a \approx \underbrace{\frac{\hbar\omega_0 \Gamma n_a}{2 I_{SAT}}}_{a_0} \frac{1}{1+s+(2\delta/\Gamma)^2} = \frac{a_0}{1+s+(2\delta/\Gamma)^2}$$

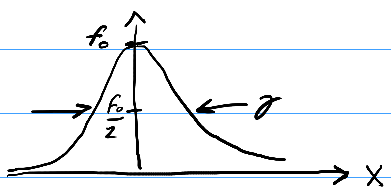
$$= \boxed{\frac{a_0}{1+s} \frac{1}{1+\left[\frac{2\delta}{\Gamma\sqrt{1+s}}\right]^2}}$$

LORENTZIAN

- GENERAL: $f(x) = \dfrac{f_0}{1+(2x/\gamma)^2}$

f_0 = MAXIMUM

γ = FWHM



ABSORPTION:

$$a(\delta) = \frac{a_0/(1+s)}{1+\left[\frac{2\delta}{\Gamma\sqrt{1+s}}\right]^2}$$

MAX: $a_0/(1+s)$

- DECREASES WITH $s \rightarrow$ "SATURATION"

FWHM: $\Gamma\sqrt{1+s}$

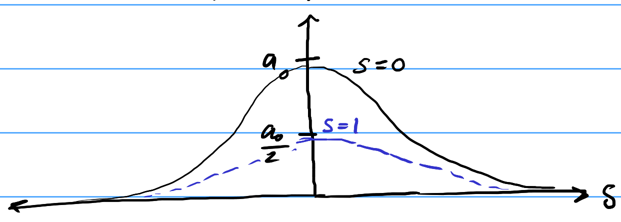
- INCREASES WITH $s \rightarrow$ "POWER BROADENING"

EX. FOR $I = I_{SAT}$, FIND: $(s=1)$

a) MAX a $= a_0/(1+s) = a_0/2$

b) FWHM $= \Gamma\sqrt{1+s} = \Gamma\sqrt{2}$

PLOT: ABSORPTION VS. δ



ABSORPTION CROSS-SECTION:

$$\sigma(\omega) = \left.\frac{a(\omega)}{n_a}\right|_{s\to 0} = \frac{a_0/n_a}{1+(2\delta/\Gamma)^2} \equiv \frac{\sigma_0}{1+(2\delta/\Gamma)^2}$$

DOPPLER BROADENING

- DOPPLER SHIFT (LAB FRAME)

$$\overset{\vec{k},\omega}{\rightsquigarrow} \quad \nearrow m\vec{v} \quad \Rightarrow \quad \nearrow m\vec{v}' = m\vec{v} + \hbar\vec{k}$$

- $$\Delta KE = \frac{1}{2}m\left(\vec{v} + \frac{\hbar}{m}\vec{k}\right)^2 - \frac{1}{2}mv^2$$

$$= \underbrace{\hbar\vec{k}\cdot\vec{v}}_{\text{"DOPPLER SHIFT"}} + \underbrace{\frac{\hbar^2k^2}{2m}}_{\text{RECOIL SHIFT (CONST) } E_r}$$

- ATOMIC STATES: $|1, m\vec{v}\rangle$, $|2, m\vec{v}+\hbar\vec{k}\rangle$
- $\Delta E = \hbar\omega_2 - \hbar\omega_1 + \hbar k\cdot v + E_r$

$$\equiv \hbar\omega_0 + \hbar\vec{k}\cdot\vec{v}$$

RESONANT FREQ: $\omega_0' = \omega_0 + \vec{k}\cdot\vec{v}$

- SUPPOSE $\omega = \omega_0$

a) $\overset{k}{\rightsquigarrow} \quad \longrightarrow v$ $\qquad k\cdot v > 0$

$\omega_0' > \omega$

$\uparrow$ ω TOO SMALL

(TOO RED)

b) $\overset{k}{\rightsquigarrow} \quad \overset{v}{\longleftarrow}$ $\qquad k\cdot v < 0$

$\omega_0' < \omega$

$\uparrow$ ω TOO LARGE (TOO BLUE)

Ex. ^{23}Na, $T = 400\,K$ $(m = 3.8\times10^{-26} kg)$

$k_B = 1.38\times10^{-23}\ J/K$

$\lambda = 589\,nm$

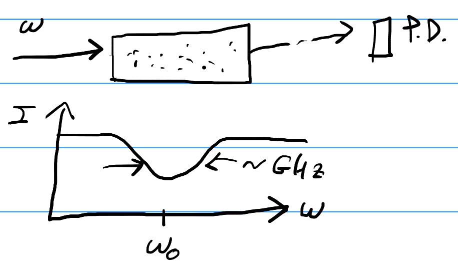
FIND RMS. DOPPLER SHIFT

MEAN SQ. VELOCITY: $\langle \frac{1}{2} m v_x^2 \rangle = \frac{1}{2} k_B T$

$$\langle v_x^2 \rangle = k_B T / m$$

$$\sqrt{\langle v_x^2 \rangle} \approx 380\ m/s$$

$$k v_{rms} = \frac{2\pi}{\lambda} v_{rms} = \frac{2\pi \times 380\ m/s}{671\ nm} = \boxed{2\pi \times 0.57\ GHz}$$



DOPPLER LINESHAPE

- $\delta = \omega - \omega_0$

CROSS-SECTION OF A MOVING ATOM:

$\Rightarrow \sigma(\delta, \vec{v}) = \sigma(\delta - \vec{k}\cdot\vec{v})$

- CONSIDER $\vec{k} = k\hat{z}$

ABSORPTION DUE TO ATOMS IN $(v_z, v_z + dv_z)$:

$$da = \sigma(\delta - k v_z) \underbrace{\frac{dn_a}{dv_z}}_{\tilde{n}_a(v_z)} dv_z$$

$$a(\omega) = \int_{-\infty}^{\infty} dv_z \underbrace{\tilde{n}_a(v_z)}_{\text{Gaussian}} \underbrace{\sigma(\delta - k v_z)}_{\text{Lorentzian}}$$ "Voigt Profile"

- Convolution

HIGH-TEMPERATURE LIMIT ($kv_{rms} \gg \Gamma$)

- LORENTZIAN → DELTA FUNCTION

NORMALIZED LORENTZIAN: $L(x) = \frac{1}{\pi}\frac{\Gamma/2}{x^2+(\Gamma/2)^2} \longrightarrow \delta(x)$ as $\Gamma \to 0$

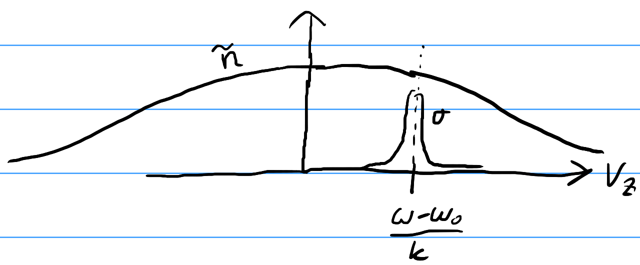
$$\Rightarrow \int_{-\infty}^{\infty} L(x)\,dx = 1$$

CROSS-SECTION : $\sigma(x) = \frac{\sigma_0}{1+\left[\frac{2}{\Gamma}x\right]^2} = \frac{\sigma_0(\Gamma/2)^2}{(\Gamma/2)^2+x^2}$

$$= \frac{\pi\sigma_0\Gamma}{2} L(x)$$

TAKE THE INTEGRAL:

$$a(\omega) = \int_{-\infty}^{\infty} dv_z \quad \tilde{n}_a(v_z)\ \sigma(\delta - kv_z) \approx \frac{\pi\sigma_0\Gamma}{2}\ \tilde{n}_a(v_z = \tfrac{\delta}{k})$$



MAXWELL-BOLTZMANN DISTRIBUTION

$$\tilde{n}(v_z) = \left[n_a\sqrt{\frac{m}{2\pi k_B T}}\ e^{-\frac{1}{2}v_z^2 \frac{m}{k_B T}} \right] \quad // \text{ DEFINE } u = \sqrt{2k_BT/m}$$

$$= \frac{n_a}{u\sqrt{\pi}}\ e^{-v_z^2/u^2}$$

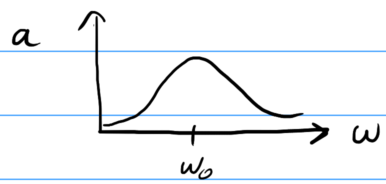
DOPPLER PROFILE

$$a(\omega) = \frac{\sigma_0\Gamma\sqrt{\pi}}{2ku}\ n_a\ e^{-(\omega-\omega_0)^2/(ku)^2}$$

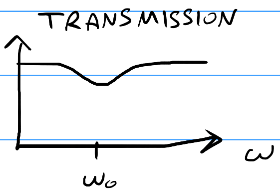
CENTER: $\omega = \omega_0$; $1/e$ WIDTH: ku

HALF WIDTH: $\frac{1}{2} = e^{-(\delta_{1/2})^2/(ku)^2} \rightarrow \ln(2) = (\delta_{1/2})^2/(ku)^2 \rightarrow \delta_{\frac{1}{2}} = ku\sqrt{\ln 2}$

$$\text{FWHM} \quad \Delta\omega_D = 2\delta_{\frac{1}{2}} = 2\sqrt{\ln 2}\, ku$$



$\Rightarrow$

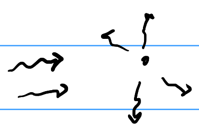


$$I(z) = I_0\, e^{-az}$$

OPTICAL FORCES ON ATOMS

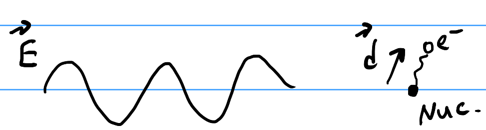
OVERVIEW

1. SCATTERING FORCE



$$\vec{F}_{sc} = \hbar\vec{k}\,\Gamma\rho_{22}$$

2. OPTICAL DIPOLE FORCE



$$U_{dipole} = -\frac{1}{2}\langle \vec{d}\cdot\vec{E}\rangle_t$$

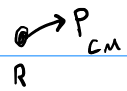
$$|\vec{d}| \propto |\vec{E}| \rightarrow U_{dipole} \propto I$$

PHY 446 SPRING 2022

TODAY: - STUDENT PRESENTATIONS

- OPTICAL FORCES ON ATOMS

ATOMIC MOTION



- FOR NOW, TREAT $\vec{R} = \vec{R}_{CM}$ AS QUANTUM VARIABLE

FORCE: $$\frac{d}{dt}\langle P_{CM}\rangle = \frac{i}{\hbar}\langle [H, P_{CM}]\rangle = \langle [H, \nabla_R]\rangle = \langle -(\nabla_R H)\rangle$$

$$= -\int \psi^*(R)(\nabla_R H)\,\psi(R)\; d^3R$$

$$H = H_0 + H'(t)$$

H_0: INDEP. OF $\vec{R}$

E1 INTERACTION:

$$H'(t) = -d\cdot E(R,t) \qquad , \qquad R = \text{ATOM C.M. POSITION}$$

$$E(R,t) = Re\left[\hat{\varepsilon}\, E_0(R)\, e^{-i\omega t + ik\cdot R}\right] = \frac{1}{2}\hat{\varepsilon}\, E_0(R)\, e^{-i\omega t + ik\cdot R} + c.c.$$

FORCE IN X-DIRECTION:

$$-\frac{\partial}{\partial x_{CM}} H' = \frac{\partial}{\partial x_{CM}}\left[d\cdot E(R,t)\right] = \vec{d}\cdot\frac{\partial}{\partial x_{CM}}\vec{E}(R,t)$$

$$= \vec{d}\cdot\left[\frac{1}{2}\hat{\varepsilon}\left(\frac{\partial E_0}{\partial x} + ik_x E_0\right)e^{-i\omega t}e^{ik\cdot R} + c.c.\right]$$

RABI FREQ:

$$\hbar\Omega e^{-i\theta} \equiv -\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle\, E_0(R)\, e^{ik\cdot R}$$

$$\hbar\Omega e^{i\theta} \equiv -\langle 1|\vec{d}\cdot\hat{\varepsilon}^*|2\rangle\, E_0(R)\, e^{-ik\cdot R}$$

θ CHOSEN S.T. $\Omega > 0$ $\quad (\Rightarrow \theta = -k\cdot R + \text{CONST})$

WAVEFUNCTION (SPINOR)

$$\psi(R,t) = a_1 e^{i(\delta t+\theta)/2 - i\omega_1 t}|1\rangle + a_2 e^{-i(\delta t+\theta)/2 - i\omega_2 t}|2\rangle$$

a_1, a_2, θ DEPEND ON $\vec{R}$

FORCE

$$\vec{F} = -\nabla_R H' = \underbrace{\frac{1}{2}(\vec{d}\cdot\hat{\varepsilon})}_{\substack{\text{ACTS ON}\\ \text{INTERNAL}\\ \text{D.O.F.}}}\underbrace{(\nabla_R E_0 + i\vec{k}E_0)\, e^{-i\omega t} e^{ik\cdot R}}_{\text{FUNCTION of } R} + h.c.$$

LET $Q = A\, f(R)$, A = 2×2 MATRIX

$$\langle Q\rangle = \int \psi^*(R)\, A\, \psi(R)\, f(R)\, d^3R \equiv \int [A]\, f(R)\, d^3R$$

$\leftarrow$ $\langle A\rangle$ OVER INTERNAL VARIABLES

$$[\vec{d}] = \psi^*(R)\, \vec{d}\, \psi(R) = a_1 a_2^* e^{i(\omega t+\theta)}\langle 2|\vec{d}|1\rangle + a_2 a_1^* e^{-i(\omega t+\theta)}\langle 1|\vec{d}|2\rangle$$

$$[\vec{d}\cdot\hat{\varepsilon}\, e^{-i\omega t} e^{ik\cdot R}] = \underbrace{a_1 a_2^*}_{\rho_{12}} e^{i\theta} e^{ik\cdot R}\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle + a_2 a_1^* \underbrace{e^{-2i\omega t}}_{\text{OSCILLATES RAPIDLY} \to \text{DROP}} e^{-i\theta} e^{ik\cdot R}\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle$$

$$[\vec{F}] = \frac{1}{2}\rho_{12}\, e^{i(\theta+k\cdot R)}\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle(\nabla_R E_0 + i\vec{k}E_0) \quad + \text{RAPIDLY OSCILLATING} \quad + \text{C.C.}$$

GRADIENT OF Ω: $\hbar\Omega(R) = -\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle E_0(R)\, e^{i\underbrace{(k\cdot R+\theta)}_{\text{CONST}}}$

$$\hbar\nabla_R\Omega = -\langle 2|\vec{d}\cdot\hat{\varepsilon}|1\rangle\, \nabla_R E_0\, e^{i(k\cdot R+\theta)}$$

$$[\vec{F}] = -\frac{1}{2}\rho_{12}(\hbar\nabla_R\Omega + i\hbar\vec{k}\Omega) \quad + \text{C.C.} + \text{RAPIDLY OSCILLATING}$$

EXPAND C.C. AND DROP RAPIDLY OSCILLATING TERMS:

$$[\vec{F}]_t = -\frac{1}{2}\rho_{12}\hbar(\nabla_R\Omega + i\vec{k}\Omega) - \frac{1}{2}\rho_{21}\hbar(\nabla_R\Omega - i\vec{k}\Omega)$$

$$= -\frac{1}{2}\hbar\vec{\nabla}_R\Omega(\rho_{12}+\rho_{21}) - \frac{1}{2}\hbar\vec{k}\Omega\; i(\rho_{12}-\rho_{21})$$

$\uparrow$ FUNCTIONS OF $\vec{R}$

SPATIAL DEGREE OF FREEDOM

- ASSUME $a_i(R) = \underset{\uparrow \text{CONST}}{a_i}\ \phi(R)$; $\rho_{ij}(R) = \rho_{ij}\,|\phi(R)|^2$
- $\phi(R)$ LOCALIZED ENOUGH THAT Ω VARIES NEGLIGIBLY OVER ATOMIC WAVEPACKET
- $\int |\phi(R)|^2 d^3R = 1$

$$\langle \vec{F} \rangle_t = \int \left[-\frac{1}{2}\hbar\,\vec{\nabla}_R \Omega (\rho_{12}+\rho_{21}) - \frac{1}{2}\hbar \vec{k}\Omega\, i(\rho_{12}-\rho_{21}) \right] |\phi(R)|^2 d^3R$$

$$= -\frac{1}{2}\hbar\,\vec{\nabla}_R \Omega (\underbrace{\rho_{12}}_{\text{INDEP. OF } \vec{R}}+\rho_{21}) - \frac{1}{2}\hbar \vec{k}\Omega\, i(\rho_{12}-\rho_{21})$$

BLOCH VECTOR

$u = \rho_{12} + \rho_{21}$

$v = i(\rho_{12} - \rho_{21})$

$w = \rho_{11} - \rho_{22}$

$$\langle \vec{F} \rangle_t = \underbrace{-\frac{1}{2}\hbar\, u\, \nabla_R \Omega}_{\text{DIPOLE FORCE}} - \underbrace{\frac{1}{2}\hbar \vec{k}\, v\, \Omega}_{\text{SCATTERING FORCE}}$$

$$= \vec{F}_{dip} + \vec{F}_{sc}$$

$$\vec{F}_{dip} = -\frac{1}{2}\hbar\, u\, \nabla_R \Omega \rightarrow -\frac{1}{2}\hbar\, \nabla_R \Omega\, \frac{\Omega\delta}{\delta^2 + \Omega^2/2 + \Gamma^2/4} = -\frac{1}{2}\hbar \left(\frac{dg}{d\Omega}\right) \nabla_R \Omega$$

$$\frac{dg}{d\Omega} = \frac{\Omega\delta}{\delta^2 + \Omega^2/2 + \Gamma^2/4}$$

$x = \Omega^2/2 + \delta^2 + \Gamma^2/4$

$$g = \int \frac{\Omega\delta}{\delta^2 + \Omega^2/2 + \Gamma^2/4} d\Omega$$

$dx = \Omega d\Omega$

$$= \int \frac{\delta}{x} dx = \delta \ln x + c = \delta \ln\left(\frac{\Omega^2/2 + \delta^2 + \Gamma^2/4}{a}\right)$$

$$0 = g(0) = \ln\left(\frac{\delta^2+\Gamma^2/4}{a}\right)$$

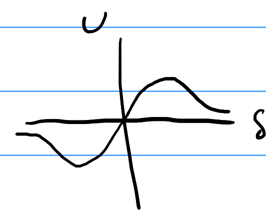
$$a = \delta^2+\Gamma^2/4$$

$$g = \delta \ln\left(1+\frac{\Omega^2/2}{\delta^2+\Gamma^2/4}\right) \approx \delta\, \frac{\Omega^2/2}{\delta^2+\Gamma^2/4}$$

DIPOLE POTENTIAL

$$\vec{F}_{dip} = -\vec{\nabla} U_{dip}$$

$$U_{dip} = \frac{\hbar}{2} g \approx \frac{\hbar\delta}{2} \frac{\Omega^2/2}{\delta^2+\Gamma^2/4}$$



SCATTERING FORCE:

$$\vec{F}_{sc} = -\frac{1}{2}\hbar\vec{k}\, v\, \Omega = \frac{1}{2}\hbar\vec{k}\, \Omega\, \frac{\Omega\Gamma/2}{\delta^2+\Omega^2/2+\Gamma^2/4}$$