The Lorentz Oscillator Model

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Contents

1 The Atom as a Harmonic Oscillator

The Lorentz oscillator model treats the atom as a classical harmonic oscillator interacting with light. This model makes accurate predictions in some situations and it will help us understand the quantum mechanical model of atom-light interactions.

1.1 Driven Oscillation of the Electron

When light shines on an atom, it causes one of the electrons to oscillate. If the light is weak enough, the oscillations will be small and we can approximate the atom as a harmonic oscillator. Experimentally, we know that atoms have discrete resonant frequencies. Suppose we want to model just one of those resonances. We can then describe the average x position of an electron relative to the nucleus using the differential equation for a driven, damped harmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = -eE_x(t)/m \tag{1}$$

Here γ is the damping rate of the oscillator, ω_0 is the angular frequency of the resonance, the electron charge is $-e$, and m is the reduced mass of the electron and the nucleus, $m \approx m_e$. The x-component of the electric field at the position of the atom is $E_x(t)$. We make the approximation $E_x(\mathbf{r}, t) \approx E_x(0, t) \equiv E_x(t)$, which is valid when the wavelength of the light is much larger than the size of the atom so that the electric field is uniform across the atom. We consider light that is monochromatic, linearly polarized in the x direction, and propagating in the z direction. The electric field of the light at the location of the atom is then:

$$E_x(t) = E_0 \cos(\omega t) = \mathrm{Re}\left[E_0 e^{-i\omega t}\right] \tag{2}$$

For convenience, we choose the phase of the oscillation such that it is represented by cosine, although the final results would be the same for any choice of phase. Note that in writing (1), we are neglecting the Lorentz force due to the magnetic field of the light, which is a factor of $\dot{z}/c$ smaller that the force due to the electric field.

The electric field (2) describes a continuous-wave (CW) light field like that produced by a CW (i.e. non-pulsed) laser. We will use the Lorentz oscillator model to understand the propagation of the light through a vapor of atoms, and to predict the forces exerted by the light on the atoms. To do so, we will employ the steady-state solution for $x(t)$, which oscillates at frequency ω. We write the steady-state solution in the form:

$$x(t) = \mathrm{Re}\left[\tilde{x} e^{-i\omega t}\right] \tag{3}$$

where $\tilde{x}$ is a complex number. The phase of $\tilde{x}$ tells us the phase of the oscillation relative to the electric field. In the language of inhomogeneous linear differential equations, (3) is a "particular solution." It is also the steady-state solution, because the homogeneous solutions to (1) decay to zero at a rate of γ. Plugging (3) into (1) gives:

$$-\omega^2 \tilde{x} - i\omega\gamma\tilde{x} + \omega_0^2 \tilde{x} = -\frac{e}{m} E_0 \tag{4}$$

$$\longrightarrow \; \tilde{x} = \frac{-eE_0/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{5}$$

The real part of $\tilde{x}$ describes the in-phase response of the oscillator, while the imaginary part represents the out-of-phase response. To see this explicitly, we write $\tilde{x}$ in terms of its real and imaginary parts:

$$\tilde{x} = \mathcal{U} - i\mathcal{V} \tag{6}$$

(the minus sign is included to be consistent with other references). The real and imaginary parts are:

$$\mathcal{U} = \frac{-eE_0}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \tag{7}$$

$$\mathcal{V} = \frac{eE_0}{m} \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \tag{8}$$

The real position $x(t)$ of the oscillator is then:

$$x(t) = \mathrm{Re}\left[(\mathcal{U} - i\mathcal{V}) e^{-i\omega t}\right] \tag{9}$$

$$= \mathcal{U}\cos(\omega t) - \mathcal{V}\sin(\omega t) \tag{10}$$

which shows that $\mathcal{U}$ is the amplitude of the in-phase response and $\mathcal{V}$ is the amplitude of the out-of-phase response (also called the in-quadrature response, because $\mathcal{U}$ and $\mathcal{V}$ can be represented as the legs of a right triangle in the complex plane).

The response can also be represented in terms of its amplitude and phase:

$$\tilde{x} = \sqrt{\mathcal{U}^2 + \mathcal{V}^2}\, e^{-i\delta} \tag{11}$$

where the phase is:

$$\delta = \cos^{-1}(\mathcal{U}/\sqrt{\mathcal{U}^2 + \mathcal{V}^2}) \tag{12}$$

The position of the oscillator in terms of the phase shift is then:

$$x(t) = \mathrm{Re}\left[\sqrt{\mathcal{U}^2 + \mathcal{V}^2}\, e^{-i(\delta + \omega t)}\right] = \sqrt{\mathcal{U}^2 + \mathcal{V}^2}\, \cos(\omega t + \delta) \tag{13}$$

1.1.1 Complex Representations

We can think of $\tilde{x}$ as the complex representation of $x(t)$. In general, given any quantity $q(t)$ that oscillates at angular frequency ω, we can define its complex amplitude $\tilde{q}$ via

$$q(t) = \mathrm{Re}\left[\tilde{q} e^{-i\omega t}\right] \tag{14}$$

The complex phase of $\tilde{q}$ encodes the phase of the oscillation. The quantity $\tilde{q}$ is often called a **phasor**.

1.2 Electric Polarization of the Atom

When the average position of the electron is displaced from the center of the atom, the atom has an **electric dipole moment**. The dipole moment will tell us a lot about how the atom interacts with light. Recall that the electric dipole moment of a collection of particles is defined as:

$$\mathbf{d} = \sum_j q_j \mathbf{r}_j \tag{15}$$

where q_j and $\mathbf{r}_j$ are the charge and position of the j-th particle. For an atom excited by laser light, the excited electron has charge $-e$ and position $\mathbf{r}_e$, while the nucleus together with all the other electrons have charge e and average position $\mathbf{r}_n$. Defining the displacement $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_n$, the dipole moment of the atom is then

$$\mathbf{d} = -e\mathbf{r}_e + e\mathbf{r}_n \tag{16}$$

$$= -e\mathbf{r} \tag{17}$$

For discussion of light that is linearly polarized in the x, or $\hat{\mathbf{i}}$, direction, the dipole moment becomes:

$$\mathbf{d}(t) = d_x(t)\,\hat{\mathbf{i}} \quad \text{with} \quad d_x(t) = -ex(t) \tag{18}$$

As with the position, we can describe the dipole moment in a complex representation:

$$d_x(t) = -e\,\mathrm{Re}\left[\tilde{x} e^{-i\omega t}\right] = \mathrm{Re}\left[\tilde{d}_x\, e^{-i\omega t}\right] \tag{19}$$

where

$$\tilde{d}_x = -e\tilde{x} \tag{20}$$

$$= \frac{e^2 E_0/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{21}$$

Equation (21) shows that the amplitude of the atomic polarization is proportional to the amplitude of the electric field. We define the **polarizability** $\alpha(\omega)$ of the atom as the proportionality constant (at the given frequency ω):

$$\tilde{d}_x = \alpha(\omega) E_0 \tag{22}$$

The polarizability is therefore:

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{23}$$

Near resonance, $\omega \approx \omega_0$, and we can approximate:

$$\alpha(\omega) \approx -\frac{e^2}{2m\omega_0}\left(\frac{1}{\Delta + i\gamma/2}\right) \tag{24}$$

where $\Delta = \omega - \omega_0$.

1.2.1 Vector Notation

For an electric field in an arbitrary direction, the complex representation is:

$$\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t) = \mathrm{Re}\left[\mathbf{E}_0 e^{-i\omega t}\right] \tag{25}$$

We define the complex representation of the dipole moment as:

$$\mathbf{d}(t) = \mathrm{Re}\left[\widetilde{\mathbf{d}}\, e^{-i\omega t}\right] \tag{26}$$

The complex dipole moment is then related to the electric field by:

$$\widetilde{\mathbf{d}} = \alpha(\omega)\mathbf{E}_0 \tag{27}$$

1.3 Polarization and Intensity of Light

So far we have assumed linearly polarized light. We can describe elliptically polarized light by using complex notation:

$$\mathbf{E}(t) = \mathrm{Re}\left[\widetilde{\mathbf{E}}\, e^{-i\omega t}\right] \tag{28}$$

$$= \mathrm{Re}[\widetilde{\mathbf{E}}]\cos(\omega t) + \mathrm{Im}[\widetilde{\mathbf{E}}]\sin(\omega t) \tag{29}$$

In that case, the complex dipole moment of the atom is:

$$\widetilde{\mathbf{d}} = \alpha(\omega)\widetilde{\mathbf{E}} \tag{30}$$

As an example, we can describe a plane wave propagating in the z direction with circular polarization in the $x - y$ plane using $\widetilde{\mathbf{E}} = \frac{1}{\sqrt{2}} E_0 e^{ikz}(1, i, 0)$.

In general, we describe the polarization of light using a complex unit vector $\hat{\boldsymbol{\varepsilon}}$ which we call the **polarization vector**:

$$\widetilde{\mathbf{E}} = E_0\, \hat{\boldsymbol{\varepsilon}}\, e^{ikz} \tag{31}$$

where E_0 is real and $\hat{\boldsymbol{\varepsilon}}^* \cdot \hat{\boldsymbol{\varepsilon}} = 1$. Some common cases:

$$\begin{array}{ll} \text{linear polarization along } z: & \hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{z}} \\ \text{right-hand circular polarization about } z: & \hat{\boldsymbol{\varepsilon}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \\ \text{left-hand circular polarization about } z: & \hat{\boldsymbol{\varepsilon}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \end{array} \tag{32}$$

For arbitrary polarization $\hat{\boldsymbol{\varepsilon}}$, the time-averaged intensity of the light is:

$$I = c\epsilon_0 \left\langle \mathbf{E} \cdot \mathbf{E} \right\rangle_t \tag{33}$$

$$= \frac{1}{2} c\epsilon_0 E_0^2 \tag{34}$$

1.4 Oscillator Strength

The classical result for the polarizability of an atom is almost correct, but quantum mechanics makes two modifications. The first is just a reminder that the harmonic oscillator approximation is only valid for weak fields. For stronger fields, we will see in the quantum treatment that the response of the atom becomes nonlinear and $\alpha(\omega)$ essentially becomes dependent on the light intensity. The second modification is that, even for weak fields, we must include a correction factor called the **oscillator strength**. For the transition from the ground state to the j-th excited state, we write the oscillator strength as $f_{0j} \geq 0$. The oscillator strength modifies the sensitivity of the atom to the electric field, and can be including by replacing $E_x \to f_{0j} E_x$ in equation (1). The polarizability due to the 0-to-j transition is then:

$$\alpha_{0j}(\omega) = \frac{f_{0j}\, e^2/m}{\omega_{j0}^2 - \omega^2 - i\gamma_j \omega} \tag{35}$$

$$\approx -\frac{e^2 f_{0j}}{2m\omega_{j0}} \left(\frac{1}{\omega - \omega_{j0} + i\gamma_j/2} \right) \tag{36}$$

Here ω_{j0} is the resonant frequency of the 0-to-j transition and γ_j is the decay rate of the j-th excited state. The total polarizability of the atom in the ground state is then given by a sum over the excited states:

$$\alpha_0(\omega) = \sum_j \alpha_{0j}(\omega) \tag{37}$$

Qualitatively, the oscillator strength accounts for the fact that the atom has many resonances. It can be loosely interpreted as the probability that the atom will behave as a harmonic oscillator with resonant frequency ω_{j0}. This interpretation is supported by the fact that the oscillator strengths sum to unity:

$$\sum_j f_{0j} = 1 \tag{38}$$

This result is known as the Thomas-Reiche-Kuhn sum rule and is proven nicely in the notes by Steck, Section 1.2, and in the book by Metcalf in appendix 3.A. It is also worth noting that the oscillator strength depends on the polarization of the light.

1.5 Radiative Damping

Classical electrodynamics predicts that an oscillating charge should radiate energy. For our model of an electron undergoing harmonic oscillation, the classical damping rate is:

$$\gamma_{cl} = \frac{e^2 \omega^2}{6\pi\epsilon_0 m_e c^3} \tag{39}$$

2 Light Propagation in an Atomic Medium

2.1 Polarization Density and Susceptiblity

If we have a gas of atoms with number density $n_a(\mathbf{R})$ and each atom near position $\mathbf{R}$ has dipole moment $\mathbf{d}(t)$, then the **polarization density** is

$$\mathbf{P} = n_a\, \mathbf{d} \tag{40}$$

In the complex representation, the complex polarization density is then:

$$\widetilde{\mathbf{P}} = n_a\, \widetilde{\mathbf{d}} = n_a\, \alpha(\omega) \widetilde{\mathbf{E}} \tag{41}$$

where we have used the general result for the dipole moment (30) in the second equation. Meanwhile, in electricity and magnetism, the complex **susceptibility** χ is defined as:

$$\widetilde{\mathbf{P}} = \epsilon_0 \chi(\omega) \widetilde{\mathbf{E}} \tag{42}$$

The susceptibility is therefore related to the polarizability by:

$$\chi(\omega) = \frac{n_a}{\epsilon_0} \alpha(\omega) \tag{43}$$

Although it's not obvious, you can check that χ is a dimensionless number. For a dilute gas (i.e. small density of atoms), $|\chi| \ll 1$.

2.2 Electromagnetic Waves

A medium, such as an atomic gas, that develops a polarization in response to an electric field is called a **dielectric** medium. The medium can also have a magnetic response, leading to a **magnetization density M**. In a moment, we will assume $\mathbf{M} = 0$, but for now we keep it. Maxwell's equations in a material are expressed with the help of **auxiliary fields D** and **H**, given by:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{44}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} + \mathbf{M} \tag{45}$$

We will assume that the free charge ρ_f and free current $\mathbf{J}_f$ are zero, meaning there are no extra charges or currents other than the atoms themselves. Maxwell's equations are then:

$$\nabla \cdot \mathbf{B} = 0 \tag{46}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{47}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{48}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{49}$$

Since we are considering a linear medium, where the polarization density is a linear response to the electric field, we can also show that the divergence of the electric field is zero:

$$\nabla \cdot \mathbf{E} = 0 \tag{50}$$

Now assuming $\mathbf{M} = 0$ for simplicity, we can also write:

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{51}$$

We can use these equations to find a wave equation for the electric field. To do so, we will use the vector calculus identity $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$ together with $\nabla \cdot \mathbf{E} = 0$ to get:

$$\nabla^2 \mathbf{E} = -\nabla \times (\nabla \times \mathbf{E}) \tag{52}$$

$$= \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \nabla \times \frac{\partial \mathbf{H}}{\partial t} \tag{53}$$

$$= \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \tag{54}$$

We will solve the wave equation for a monochromatic field of the form:

$$\mathbf{E} = \mathrm{Re}\left[\widetilde{\mathbf{E}}\, e^{-i\omega t}\right] \tag{55}$$

where $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}(\mathbf{r}')$ is a function of position $\mathbf{r}'$. Using the definition (42) of $\chi(\omega)$ the polarization density is:

$$\mathbf{P} = \mathrm{Re}\left[\epsilon_0 \chi(\omega) \widetilde{\mathbf{E}}\, e^{-i\omega t}\right] \tag{56}$$

The definition (44) of the "displacement field" **D** gives:

$$\mathbf{D} = \mathrm{Re}\left[\epsilon_0 (1 + \chi)\, \widetilde{\mathbf{E}}\, e^{-i\omega t}\right] \tag{57}$$

Substituting the equations (55) and (57) for **E** and **D** in the wave equation (54) gives:

$$\nabla^2 \widetilde{\mathbf{E}} = -k_0^2 (1 + \chi) \widetilde{\mathbf{E}} \tag{58}$$

where we have introduced the definition

$$k_0 = \omega/c \tag{59}$$

and c is the speed of light in vacuum:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{60}$$

We can solve the differential equation (58) for $\widetilde{\mathbf{E}}$ for a plane wave traveling in the z direction:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 \, e^{i\tilde{k}z} \tag{61}$$

where $\tilde{k}$ is a complex number. Plugging our plane wave (61) into the differential equation (58) for $\widetilde{\mathbf{E}}$ gives:

$$\tilde{k} = k_0\sqrt{1+\chi} \tag{62}$$

Here we have chosen the positive square root to describe motion in the $+z$ direction. Since χ is complex, $\sqrt{1+\chi}$ is a complex number. We call it the **complex index of refraction** $\tilde{n}$:

$$\begin{aligned} \tilde{n} &= \sqrt{1+\chi} && (63) \\ &\approx 1 + \frac{1}{2}\chi && (64) \end{aligned}$$

where the second line uses the Taylor expansion for $|\chi| \ll 1$, valid for a dilute gas. Formally, we have now solved for the electric field of a plane wave in an atom medium. Collecting the above results, we can write our solution as:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 \, e^{i\tilde{n}k_0 z} \tag{65}$$

In the next section we will study the physical meaning of this solution. We will see that the real part of $\tilde{n}$ corresponds to the usual index of refraction and leads to a phase shift of the light, while the imaginary part of $\tilde{n}$ leads to absorption of the light.

2.3 Phase Shift and Absorption

We separate $\tilde{n}$ into real and imaginary parts:

$$n_r = \mathrm{Re}[\tilde{n}] \approx 1 + \frac{1}{2}\mathrm{Re}[\chi] \tag{66}$$

$$n_i = \mathrm{Im}[\tilde{n}] \approx \frac{1}{2}\mathrm{Im}[\chi] \tag{67}$$

The complex electric field then propagates according to:

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}_0 \, e^{in_r k_0 z} \, e^{-n_i k_0 z} \tag{68}$$

The wavevector is increased by a factor of n_r compared to the vacuum wavevector $k_0 = \omega/c$. Therefore, n_r is the ordinary **index of refraction**, also called the **phase index**. After a distance z, the phase of the light wave will differ from what it would have been in vacuum by an amount:

$$\Delta\phi = (n_r - 1)k_0 z \tag{69}$$

The phase shift can be detected by measuring shifts of interference fringes in an interferometer.

The imaginary part n_i causes absorption. The intensity $I(z)$ of the light is proportional to $|\widetilde{\mathbf{E}}|^2$, so the intensity decays exponentially:

$$\begin{aligned} I(z) &= I_0 e^{-2n_i k_0 z} && (70) \\ &= I_0 e^{-az} && (71) \end{aligned}$$

In the second line above we have introduced the **absorption coefficient** a:

$$a(\omega) = 2n_i k_0 \tag{72}$$

2.3.1 Phase Velocity and Group Velocity

Writing the effective wavevector as $k = n_r k_0$, we can obtain the **phase velocity** as:

$$v_p = \frac{\omega}{k} = \frac{c}{n_r} \tag{73}$$

The phase velocity gives the speed at which the phase fronts of the wave travel. If $n_r < 1$, the phase velocity would exceed c. Can this happen? Let's see! For a dilute gas in the Lorentz oscillator model, the phase index is:

$$n_r \approx 1 + \frac{n_a e^2}{2\epsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{74}$$

When $\omega > \omega_0$, the second term is negative and we have $n_r < 1$! However, the phase velocity is an artificial quantity, and it does not represent the speed of information travel, so there is no conflict with special relativity.

To find the speed of information travel, we need to look at the speed of a pulse, or wave packet. This is called the **group velocity** and is given by:

$$v_g = \frac{1}{dk/d\omega} = c \left[\frac{d(n_r\omega)}{d\omega} \right]^{-1} \tag{75}$$

As it turns out, $v_g \leq c$, as required by relativity.

2.3.2 Absorption Cross Section

For a dilute gas, we can describe the absorption of light using the concept of an **absorption cross section**. Imagine that each atom is actually an opaque object with a cross-sectional area of σ. As light propagates, it would then be attenuated according to:

$$\frac{dI}{dz} = -n_a \sigma I \tag{76}$$

This is called **Beer's law** of absorption or the **Beer-Lambert** law. Comparing to our earlier result (70) for light absorption, we see:

$$\sigma(\omega) = \frac{\omega}{\epsilon_0 c} \mathrm{Im}[\alpha(\omega)] \tag{77}$$

where we have used the dilute gas approximation of (67).

3 Optical Forces on Atoms

The electric dipole moment of an atom interacts with light, leading to an potential energy U. We will see that this potential energy is proportional to the light intensity in the Lorentz oscillator model. According to classical mechanics, a gradient in potential energy leads to a force through $\mathbf{F} = -\nabla U$. Therefore, a gradient in light intensity will cause a force to be exerted on an atom. This force is used to trap atoms and other polarizable particles using a technique called **optical tweezers** or **optical dipole trapping**.

3.1 DC Electric Field

3.1.1 Potential Energy

As a warm-up, let's first consider a *static* electric field $\mathbf{E} = E\,\hat{\mathbf{i}}$. Consider a particle with DC polarizability $\alpha = \alpha(0)$, so that its dipole moment is $d_x = -ex = \alpha E$. Note that α is purely real at $\omega = 0$. Increasing the electric field by an amount dE stores an energy dU in the system, given by the Work-Energy Theorem from classical mechanics:

$$dU = -F dx = -(-eE)dx = -E\,d(-ex) = -E\,d(\alpha E) = -\alpha E\,dE \tag{78}$$

The potential energy of a polarizable particle in a static electric field of strength E is then:

$$U = -\alpha \int_0^E E'\,dE' = -\frac{1}{2}\alpha E^2 \tag{79}$$

The electric field could have been chosen to point in any direction, so in general we can interpret the E^2 in (79) as the square of the magnitude of the field, $E^2 = \mathbf{E}\cdot\mathbf{E}$. Since the dipole moment is $\mathbf{d} = \alpha\mathbf{E}$, we can also write this result as:

$$U = -\frac{1}{2}\mathbf{d}\cdot\mathbf{E} \tag{80}$$

As you can see from the above argument, the factor of $\frac{1}{2}$ in equation (80) results from the fact that the dipole is *induced* by the field. In contrast, a particle with a permanent dipole moment simply has a potential energy $-\mathbf{d}\cdot\mathbf{E}$.

In the above derivation, we have implicitly assumed that the electric field is uniform. Specifically, we assumed that the electric field is the same at the center of the atom as it is at the position of the electron. To see this, recall that, rigorously speaking, x is actually the displacement of the electron from the rest of the atom, $x = x_e - x_n$. The work done on the atom by increasing the field is then proportional to $E_e dx_e - E_n dx_n$, where $E_e = E(x_e)$ and $E_n = E(x_n)$. By assuming $E_e = E_n = E$, we can factor out the E and get $E\,dx$ as in equation (78). For a non-uniform electric field, the final result in equations (79) and (80) is still accurate as long as the electric field varies by only a small amount over the size of the atom.

3.1.2 Force in a Non-Uniform Field

Since an atom is neutral, a uniform electric field exerts no net force on the center of mass of the atom. However, if the electric field varies with position, it will exert a non-zero force on the atom. The force is given by:

$$\mathbf{F} = -\nabla U = \frac{1}{2}\alpha\nabla(E^2) \tag{81}$$

At first, (81) may seem counter-intuitive: it says that the direction of the force is along the gradient of E^2. But what if $\mathbf{E}$ points in the x direction, while its magnitude changes along the y direction? The equation $\mathbf{F} = q\mathbf{E}$ for the force on a charge suggests that the net force can only be along the direction of $\mathbf{E}$, i.e. the x direction in this example. How can the net force be in the y direction? The resolution of this paradox lies in Maxwell's equations. In vacuum, a static electric field satisfies $\nabla\cdot\mathbf{E} = 0$ and $\nabla\times\mathbf{E} = 0$. Therefore, if the x-component E_x of the electric field varies along y, the field *must* have a non-zero y component. Specifically, from the curl equation, $\partial E_y/\partial x = \partial E_x/\partial y \neq 0$, which means that E_y cannot be zero everywhere. Since the field has a y component, it is able to exert a force in the y direction. Equation (81) conveniently does not depend on the direction of $\mathbf{E}$, so you can use it if you just know the magnitude of the field.

The equation for the force on a polarizable particle in a non-uniform static electric field can also be derived by considering the forces on the individual charges. Consider an atom with center of mass position $\mathbf{R} = 0$, its nucleus (and all but one of the electrons) centered at $\mathbf{r}_n$, and one of its electrons displaced to the average position $\mathbf{r}_e$. The i-th component of the net force on the atom is:

$$\begin{aligned}
F_i &= -e\,E_i(\mathbf{r}_e) + e\,E_i(\mathbf{r}_n) && (82)\\
&\approx -e(\mathbf{r}_e\cdot\nabla)E_i|_{\mathbf{R}=0} + e(\mathbf{r}_n\cdot\nabla)E_i|_{\mathbf{R}=0} && (83)\\
&= -e\left[(\mathbf{r}_e - \mathbf{r}_n)\cdot\nabla\right]E_i|_{\mathbf{R}=0} && (84)\\
&\equiv -e(\mathbf{r}\cdot\nabla)E_i|_{\mathbf{R}=0} && (85)
\end{aligned}$$

where $E_i(\mathbf{r}_e)$ and $E_i(\mathbf{r}_n)$ have been Taylor expanded about $\mathbf{R} = 0$ in the second line, and we have defined $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_n$. The Taylor expansion makes it clear that we are assuming the field varies by a small amount over the size of the atom. Moving to vector notation for $\mathbf{F}$, and leaving the $\mathbf{R} = 0$ implicit for simplicity, we have:

$$\mathbf{F} = -e(\mathbf{r} \cdot \nabla)\mathbf{E} = \alpha(\mathbf{E} \cdot \nabla)\mathbf{E} \tag{86}$$

Here we have used the fact that the dipole moment is $-e\mathbf{r} = \mathbf{d} = \alpha\mathbf{E}$. Finally, we need to use a vector identity:

$$(\mathbf{E} \cdot \nabla)\mathbf{E} = \frac{1}{2}\nabla(E^2) - \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{87}$$

Since the field is static, $\nabla \times \mathbf{E} = 0$ and we finally obtain:

$$\mathbf{F} = \frac{1}{2}\alpha\nabla(E^2) \tag{88}$$

This shows that we get the same net force whether we start from the potential energy or from the forces on the individual particles.

3.2 Optical Forces in an AC Field

Now we derive the force on a polarizable particle in a non-uniform electromagnetic field that oscillates at angular frequency ω. We will use the method of calculating the total force on the individual charges. First, since we've seen that a non-uniform field cannot point purely in the x-direction in general, let's write the electric field at position $\mathbf{R}$ in vector notation:

$$\mathbf{E}(\mathbf{R}, t) = \mathbf{E}_0(\mathbf{R}) \cos\left[\omega t - \phi(\mathbf{R})\right] \tag{89}$$

Here $\mathbf{E}_0(\mathbf{R})$ is assumed to vary slowly with $\mathbf{R}$. The phase $\phi(\mathbf{R})$ describes the propagation of the light. For light with wavevector $\mathbf{k}$, $\phi(\mathbf{R}) \approx \mathbf{k} \cdot \mathbf{R}$. In addition to the $\mathbf{k} \cdot \mathbf{R}$ term in the phase, there is also a contribution called the Gouy phase, however the exact form will not be important here. In addition to the electric field, Maxwell's equations require that the electromagnetic wave also has a non-zero magnetic field $\mathbf{B}(\mathbf{R}, t) \approx \mathbf{B}_0(\mathbf{R}) \cos\left[\omega t - \phi(\mathbf{R})\right]$, with $\mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0/c$.

To find the force on the atom, we must include the Lorentz force due the magnetic field:

$$\begin{aligned}\mathbf{F} &= -e\mathbf{E}_e + e\mathbf{E}_n - e\frac{d\mathbf{r}_e}{dt} \times \mathbf{B}_e + e\frac{d\mathbf{r}_n}{dt} \times \mathbf{B}_n && (90)\\ &\approx (-e\mathbf{r} \cdot \nabla)\mathbf{E} - e\frac{d\mathbf{r}}{dt} \times \mathbf{B} && (91)\end{aligned}$$

In the first line, we used the abbreviations $\mathbf{E}_e = \mathbf{E}(\mathbf{r}_e)$, etc. In the second line, we have taken the leading term in the Taylor expansions in $\mathbf{r}_e$ and $\mathbf{r}_n$, similar to what we did for the static field case in equation (83). As before, we defined $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_n$. We can now substitute in expressions for $-e\mathbf{r}$ from our treatment of the Lorentz oscillator:

$$\begin{aligned}-e\mathbf{r} = \mathbf{d} &= \mathrm{Re}\left[\alpha(\omega)\mathbf{E}_0\, e^{-i(\omega t - \phi)}\right] && (92)\\ &= \mathrm{Re}\left[\alpha(\omega)\right]\mathbf{E}_0 \cos(\omega t - \phi) + \mathrm{Im}\left[\alpha(\omega)\right]\mathbf{E}_0 \sin(\omega t - \phi) && (93)\\ &= \alpha_r(\omega)\mathbf{E} + \alpha_i(\omega)\mathbf{E}_0 \sin(\omega t - \phi) && (94)\end{aligned}$$

The first line is just a generalization of equations (26) and (27) to include the phase. In the last time, we introduced the notation $\alpha_r(\omega) = \mathrm{Re}\left[\alpha(\omega)\right]$ and $\alpha_i(\omega) = \mathrm{Im}\left[\alpha(\omega)\right]$. For the time derivative, we get:

$$\begin{aligned}-e\frac{d\mathbf{r}}{dt} &= \alpha_r(\omega)\frac{\partial \mathbf{E}}{\partial t} + \omega\, \alpha_i(\omega)\mathbf{E}_0 \cos(\omega t - \phi) && (95)\\ &= \alpha_r(\omega)\frac{\partial \mathbf{E}}{\partial t} + \omega\, \alpha_i(\omega)\mathbf{E} && (96)\end{aligned}$$

In the above, we have assumed that the velocity of the atom is initially zero, so that $\frac{d}{dt}\mathbf{E}(\mathbf{R}(t), t) = \partial\mathbf{E}/\partial t$.

When we calculate the force, we will average it over a time that is large compared to the period $2\pi/\omega$ of the light. We use the following property of time averages of sinusoidal functions:

$$\langle\cos(\omega t)\sin(\omega t)\rangle_t = 0 \tag{97}$$

The time-averaged force is then:

$$\langle\mathbf{F}\rangle_t = \left\langle(-e\mathbf{r}\cdot\nabla)\mathbf{E} - e\frac{d\mathbf{r}}{dt}\times\mathbf{B}\right\rangle_t \tag{98}$$

$$= \left\langle\alpha_r(\omega)(\mathbf{E}\cdot\nabla)\mathbf{E} + \alpha_r(\omega)\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \omega\,\alpha_i(\omega)\mathbf{E}\times\mathbf{B}\right\rangle_t \tag{99}$$

In the above, we used (97) to eliminate the second term coming from (94). We can now use the vector identity (87) along with the Maxwell equation $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ to get:

$$\langle\mathbf{F}\rangle_t = \left\langle\alpha_r(\omega)\left[\frac{1}{2}\nabla(E^2) + \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})\right] + \omega\,\alpha_i(\omega)\mathbf{E}\times\mathbf{B}\right\rangle_t \tag{100}$$

The quantity $\mathbf{E}\times\mathbf{B}$ is proportional to the Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$. The Poynting vector gives the flux of energy carried by the electromagnetic wave. Assuming a steady (CW) laser beam, the time derivative of $\mathbf{S}$ will average to zero, allowing us to drop the middle term in (100). Meanwhile, the time-average of the Poynting vector is related to the light intensity I and the direction of propagation $\hat{\mathbf{k}}$ of the light wave:

$$\langle\mathbf{S}\rangle = I\hat{\mathbf{k}} \tag{101}$$

The time-average of E^2 is also proportional to I:

$$\left\langle E^2\right\rangle = \frac{I}{\epsilon_0 c} \tag{102}$$

The time-averaged force can then be written as a sum of two terms:

$$\langle\mathbf{F}\rangle_t = \mathbf{F}_{\text{dipole}} + \mathbf{F}_{\text{scatt}} \tag{103}$$

where the first term is due to the real part of $\alpha(\omega)$:

$$\mathbf{F}_{\text{dipole}} = \frac{\alpha_r(\omega)}{2\epsilon_0 c}\nabla I \tag{104}$$

and the second term is due to the imaginary part of $\alpha(\omega)$:

$$\mathbf{F}_{\text{scatt}} = \omega\alpha_i(\omega)\mu_0 I\,\hat{\mathbf{k}} \tag{105}$$

3.3 Dipole Potential

The "dipole" force $\mathbf{F}_{\text{dipole}}$ can be interpreted as resulting from the potential energy of the induced atomic dipole in the electric field of the light. To see this, we note that $\mathbf{F}_{\text{dipole}}$ can be written as the gradient of a potential function:

$$\mathbf{F}_{\text{dipole}} = \nabla\left\langle\frac{1}{2}\alpha_r(\omega)E^2\right\rangle_t \equiv -\nabla U_{\text{dipole}} \tag{106}$$

So the dipole potential is:

$$U_{\text{dipole}} = -\frac{1}{2}\alpha_r(\omega)\left\langle E^2\right\rangle_t \tag{107}$$

$$= -\frac{\alpha_r(\omega)}{2\epsilon_0 c}I \tag{108}$$

On the other hand, the time average of $\mathbf{d} \cdot \mathbf{E}$ is:

$$\langle \mathbf{d} \cdot \mathbf{E} \rangle_t = \langle \alpha_r(\omega) \mathbf{E} \cdot \mathbf{E} \rangle_t = \alpha_r(\omega) \left\langle E^2 \right\rangle_t \tag{109}$$

So we can also write the dipole potential as:

$$U_{\text{dipole}} = -\frac{1}{2} \langle \mathbf{d} \cdot \mathbf{E} \rangle_t \tag{110}$$

Equation (110) is the time average of the equation for the potential energy of an induced DC dipole (80), which makes a nice connection between the DC and AC cases. In practice, equation (108) is the most useful form of the dipole potential here, because it involves the light intensity, which is usually measured in experiments.

3.4 Radiation Pressure Force

The "scattering" force $\mathbf{F}_{\text{scatt}}$ results from the momentum transfered to the atom as it scatters light from the laser beam. This force is also referred to as **radiation pressure** because it is exerted in the direction of the light propagation $\hat{\mathbf{k}}$. The scattering force is not a conservative force in the sense that it cannot generally be written as the gradient of a potential energy. To see this, you can check that the curl of the force is non-zero:

$$\nabla \times \mathbf{F}_{\text{scatt}} = [\omega \alpha_i(\omega) \mu_0 \nabla I] \times \hat{\mathbf{k}} \neq 0 \tag{111}$$

To see that this is non-zero, note that the gradient of the intensity of a laser beam points mostly in the transverse direction, while $\hat{\mathbf{k}}$ points in the longitudinal direction, so their cross product is non-zero. Since $\nabla \times \mathbf{F}_{\text{scatt}}$ is non-zero, the vector field $\mathbf{F}_{\text{scatt}}$ cannot be written as the gradient of a scalar function.

Since the scattering force is not conservative, it can **dissipate** energy from the system. This fact is exploited in **laser cooling** to cool gases of atoms or other particles to near absolute zero temperature. In laser cooling, energy from the atomic motion is irreversibly transferred to the electromagnetic field through light scattering. On the other hand, light scattering can also lead to heating, depending on the situation.

PHY 446 Lecture 3

- Absorption & Phase Shift of light by atoms

WARM-UP SIMPLIFY THE FOLLOWING, INTERPRET

a) $\vec{E}(z,t) = \mathrm{Re}\left[E_0 \hat{x}\, e^{ikz} e^{-i\omega t}\right]$

$$= E_0 \hat{x} \cos(\omega t - kz)$$

LINEARLY POLARIZED ALONG $\hat{x}$

PROPAGATING ALONG z

b) $\vec{E}(z,t) = \mathrm{Re}\left[E_0 \hat{x}\, e^{-\alpha z} e^{ikz} e^{-i\omega t}\right]$

$$= E_0 \hat{x}\, e^{-\alpha z} \cos(\omega t - kz)$$

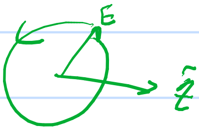
ATTENUATED EXPONENTIALLY

c) $\vec{E}(z,t) = \mathrm{Re}\left[E_0 \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})\, e^{ikz} e^{-i\omega t}\right]$

$$= \frac{E_0}{\sqrt{2}}\left(\hat{x}\cos(\omega t - kz) + \hat{y}\sin(\omega t - kz)\right)$$

CIRCULAR POLARIZATION

CCW ABOUT $\hat{z}$ AT FIXED $\vec{r}$



GENERALIZE - PLANE WAVE ALONG z

$$\vec{E}(z,t) = \mathrm{Re}\left[\underset{\uparrow\ \text{vector}}{\tilde{\vec{E}}(z)}\, e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[\underset{\substack{\uparrow\\ \text{real}\\ \text{scaler}}}{E_0}\ \underset{\substack{\uparrow\\ \text{Polarization}\\ \text{vector}}}{\hat{\varepsilon}}\ \underset{\substack{\uparrow\\ z\text{-dependence}}}{e^{(ik-\alpha)z}}\quad e^{-i\omega t}\right]$$

NOTE: $\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}$ for PLANE WAVES

NORMALIZATION: $\hat{\varepsilon}^{*}\cdot\hat{\varepsilon} = 1$

i.e. for $\hat{\varepsilon} = \frac{1}{\sqrt{2}}(1, i, 0) = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$

$$\hat{\varepsilon}^{*} = \frac{1}{\sqrt{2}}(1, -i, 0)$$

$$\hat{\varepsilon}\cdot\tilde{\varepsilon}^{*} = \frac{1}{2} + \frac{1}{2} = 1$$

SUSCEPTIBILITY

ELECTRIC FIELD: $\vec{E}(\vec{r},t) = \mathrm{Re}\left[\tilde{E}(\vec{r})\, e^{-i\omega t}\right]$

POLARIZATION DENSITY: $\vec{P}(\vec{r},t) = \mathrm{Re}\left[\tilde{P}(\vec{r})\, e^{-i\omega t}\right]$

- DUE TO i.e. ATOMIC DIPOLES

SUSCEPTIBILITY: $\chi(\omega)$

$$\tilde{P} = \varepsilon_0 \chi(\omega) \tilde{E}$$

ATOMIC VAPOR:

WE SHOWED: $\chi(\omega) = \frac{n_a}{\varepsilon_0} \alpha(\omega)$

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

DC LIMIT:

FIND $\lim_{\omega \to 0} \chi(\omega) = \frac{e^2}{m\omega_0^2} \frac{n_a}{\varepsilon_0} \equiv \chi_0$

- REAL

FOR $\omega \ll \omega_0$: $\vec{P}(t) \approx \mathrm{Re}\left[\varepsilon_0 \chi_0 \tilde{E}\, e^{-i\omega t}\right]$

$$= \varepsilon_0 \chi_0 \vec{E}(t)$$

- APPLICABLE TO GLASS IN VISIBLE

LIGHT PROPAGATION

PLANE WAVE TRAVELING IN $\hat{z}$ DIRECTION

FROM MAXWELL'S EQNS:

$$\tilde{E}(r) = E_0 \hat{\varepsilon} e^{i \tilde{n} k_0 z}$$

k_0 = "VACUUM WAVEVECTOR" = ω / c

$\tilde{n}$ = "COMPLEX INDEX OF REFRACTION"

$$= \sqrt{1 + \chi(\omega)} \qquad \text{(From MAXWELL)}$$

$$\approx 1 + \tfrac{1}{2}\chi(\omega) \quad \text{for } |\chi(\omega)| << 1$$

$$= 1 + \frac{n_a}{2\varepsilon_0} \frac{e^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$= 1 + \frac{n_a e^2}{2\varepsilon_0 m} \frac{(\omega_0^2 - \omega^2) + i\gamma\omega}{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

REAL AND IMAG. PARTS:

$$\tilde{n} \equiv n_r + i n_i$$

$$n_r \approx 1 + \frac{n_a e^2}{2\varepsilon_0 m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

$$n_i \approx \frac{n_a e^2}{2\varepsilon_0 M} \frac{\gamma\omega}{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

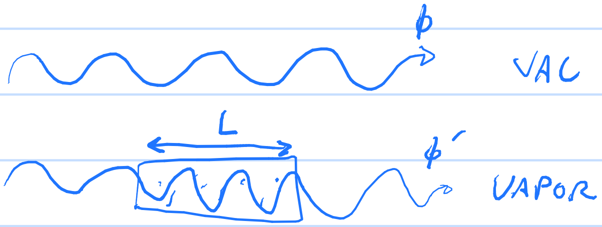
ABSORPTION & PHASE SHIFT

$$\tilde{E}(\vec{r}) = E_0 \hat{\varepsilon}\, e^{i\tilde{n}k_0 z} = E_0 \hat{\varepsilon}\, e^{-n_i k_0 z}\, e^{i n_r k_0 z}$$

REAL PART: PHASE

$$\phi(z) = n_r k_0 z \equiv k z$$

COMPARE:



$$\Delta\phi = n_r k_0 L - k_0 L$$
$$= (n_r - 1) k_0 L$$

- CAN MEASURE VIA INTERFEROMETER
  OR REFRACTION (BENDING) OF LIGHT
  AT NON-NORMAL INCIDENCE

IMAG. PART $\tilde{E} \propto e^{-n_i k_0 z}$

LIGHT INTENSITY:

$$I = \frac{\text{energy}}{\text{area}\cdot\text{time}} = \frac{c}{n_r}\bar{u}$$

energy density (time avg.)

$$= \frac{c}{n_r}\frac{n_r^2}{2}\left(\varepsilon_0 \overline{E^2} + \frac{1}{\mu_0}\overline{B^2}\right)$$

$$= \varepsilon_0 c\, \overline{E^2}\, n_r \text{ for plane waves}$$

$$= \frac{n_r}{2}\varepsilon_0 c\, \tilde{E}\cdot\tilde{E}^* = \underbrace{\frac{n_r}{2}\varepsilon_0 c\, E_0^2}_{I_0}\, e^{-2n_i k_0 z}$$

$$I(z) = I_0 e^{-2 n_i k_0 z}$$

$$\equiv I_0 e^{-a z}$$

$$a(\omega) = 2 n_i k_0$$

$$= \frac{2\omega}{c} \, \mathrm{Im}(\tilde{n}) \approx \frac{2\omega}{c} \frac{n_a e^2}{2 \varepsilon_0 m_e} \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + (\gamma \omega)^2}$$

Exercise: Relate to $\mathrm{Im}\, \alpha(\omega)$

LIGHT INTENSITY - NOTE

DERIVATION OF INTENSITY USING POYNTING VECTOR

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\vec{E} = \mathrm{Re}\left[E_0 \hat{\varepsilon}\, e^{i\tilde{k}z}\, e^{-i\omega t}\right]$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \left(\cancel{\partial_y E_z} - \partial_z E_y,\ \partial_z E_x - \cancel{\partial_x E_z},\ \cancel{\partial_x E_y} - \cancel{\partial_y E_x}\right)$$

$$= \left(-\partial_z E_y,\ \partial_z E_x,\ 0\right)$$

$$\partial_z \vec{E} = \mathrm{Re}\left[E_0 \hat{\varepsilon}\ i\tilde{k}\ e^{i\tilde{k}z}\, e^{-i\omega t}\right]$$

$$\vec{B} = \mathrm{Re}\left[B_0 \hat{b}\, e^{i\tilde{k}z}\, e^{-i\omega t}\right]$$

$$-\frac{\partial \vec{B}}{\partial t} = \mathrm{Re}\left[B_0 \hat{b}\, i\omega\, e^{i\tilde{k}z - i\omega t}\right]$$

$\hat{x}$ component:

$$(\nabla \times E)_x = -\partial_z E_y = -\mathrm{Re}\left[E_0 \varepsilon_y\, i\tilde{k}\, e^{i(\tilde{k}z - \omega t)}\right]$$

$$-\partial B_x / \partial t = \mathrm{Re}\left[B_0 \hat{b}_x\, i\omega\, e^{i\tilde{k}z - i\omega t}\right]$$

$$B_0 b_x \omega = -E_0 \varepsilon_y \tilde{k}$$

$\hat{y}$: $(\nabla \times E)_y = \partial_z E_x = \mathrm{Re}\left[E_0 \varepsilon_x\, i\tilde{k}\, e^{i(\tilde{k}z - \omega t)}\right]$

$$-\partial B_y / \partial t = \mathrm{Re}\left[B_0 b_y\, i\omega\, e^{i(\tilde{k}z - \omega t)}\right]$$

$$B_0 b_y \omega = E_0 \varepsilon_x \tilde{k}$$

$$b_x = -\varepsilon_y \frac{\tilde{k}}{|\tilde{k}|} \; ; \; b_y = \varepsilon_x \frac{\tilde{k}}{|\tilde{k}|}$$

$$b_x^* b_x + b_y^* b_y = \varepsilon_y^* \varepsilon_y + \varepsilon_x^* \varepsilon_x = 1$$

$$B_0^2 b_x^* b_x \omega^2 + B_0^2 \omega^2 b_y^* b_y = E_0^2 \varepsilon_y^* \varepsilon_y |\tilde{k}|^2 + E_0^2 \varepsilon_x^* \varepsilon_x |\tilde{k}|^2$$

$$B_0^2 \omega^2 = E_0^2 |\tilde{k}|^2$$

$$B_0 = \frac{|\tilde{k}|}{\omega} E_0$$

$$\vec{B} = \mathrm{Re}\left[B_0 \hat{b} e^{i\tilde{k}z} e^{-i\omega t} \right]$$

$$= \frac{1}{2}\left[B_0 \hat{b} e^{i(\tilde{k}z-\omega t)} + B_0 \hat{b}^* e^{-i(\tilde{k}z-\omega t)} \right]$$

$$\vec{E} = \mathrm{Re}\left[E_0 \hat{\varepsilon} e^{i\tilde{k}z} e^{-i\omega t} \right]$$

$$= \frac{1}{2}\left[E_0 \hat{\varepsilon} e^{i(\tilde{k}z-\omega t)} + E_0 \hat{\varepsilon}^* e^{-i(\tilde{k}z-\omega t)} \right]$$

$$\hat{\varepsilon} \times \hat{b}^* = \left(0, 0, \varepsilon_x b_y^* - \varepsilon_y b_x^* \right)$$

$$= \varepsilon_x^* \varepsilon_x \frac{\tilde{k}^*}{|\tilde{k}|} + \varepsilon_y^* \varepsilon_y \frac{\tilde{k}^*}{|\tilde{k}|} = \frac{\tilde{k}^*}{|\tilde{k}|}$$

$$\hat{\varepsilon}^* \times \hat{b} = \tilde{k}/|\tilde{k}|$$

$$\langle \vec{E} \times \vec{B} \rangle_t = \frac{1}{4} E_0 B_0 \left(\frac{\tilde{k}^*}{|\tilde{k}|} + \frac{\tilde{k}}{|\tilde{k}|} \right) = \frac{1}{2} E_0 B_0 \frac{\mathrm{Re}(\tilde{k})}{|\tilde{k}|}$$

$$= \frac{1}{2} E_0 \left(\frac{\cancel{|\tilde{k}|}}{\cancel{\omega}} E_0 \right) \left(n_r \frac{\cancel{\omega}}{c} \right) \frac{1}{\cancel{|\tilde{k}|}}$$

$$= \frac{1}{2} \left(\frac{n_r}{c} \right) E_0^2 = \frac{1}{2} \sqrt{\varepsilon_0 \mu_0}\, n_r E_0^2$$

$$\langle \vec{S} \rangle_t = \frac{1}{\mu_0} \langle \vec{E} \times \vec{B} \rangle_t = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}}\, n_r E_0^2 \hat{z}$$

$$= \frac{1}{2} \frac{\varepsilon_0}{\sqrt{\varepsilon_0 \mu_0}} n_r E_0^2 \hat{z} = \boxed{\frac{1}{2} \varepsilon_0 c\, n_r E_0^2 \hat{z}}$$

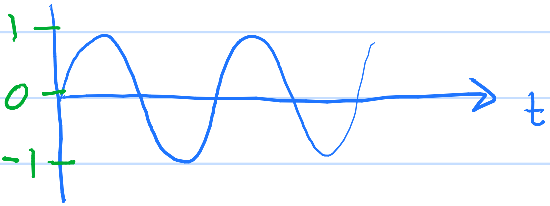
PHY 446 LECTURE 4

- OPTICAL FORCES ON ATOMS

WARM-UP 1

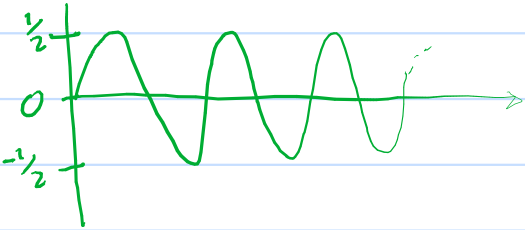
SKETCH THE FOLLOWING AND FIND AVG VALUE:

a) $\sin(\omega t)$



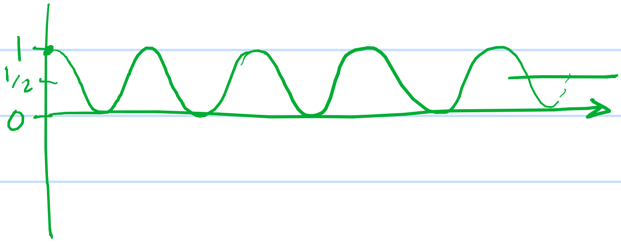
$\leftarrow$ AVG = 0

b) $\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$



$\leftarrow$ AVG = 0

c) $\cos^2(\omega t) = \frac{1}{2}(1+\cos(2\omega t))$



AVG = $\frac{1}{2}$

TIME AVERAGE FORMALLY,

$$\langle f(t) \rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)\, dt$$

E. G.

$$\langle \cos(\omega t) \rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos(\omega t)\, dt$$

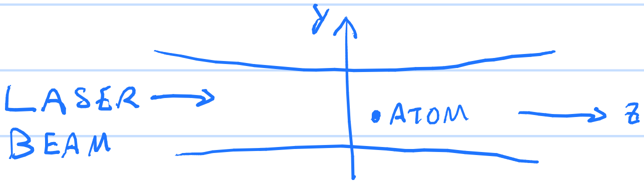
$$= \lim_{T \to \infty} \frac{\sin(\omega T)}{\omega T} = 0$$

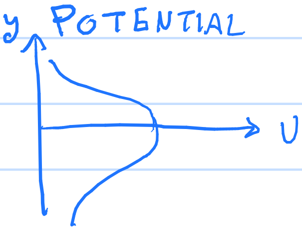
WE'LL USE THIS IN A FEW MINUTES...

TODAY: OPTICAL FORCES ON ATOMS

1) DIPOLE FORCE

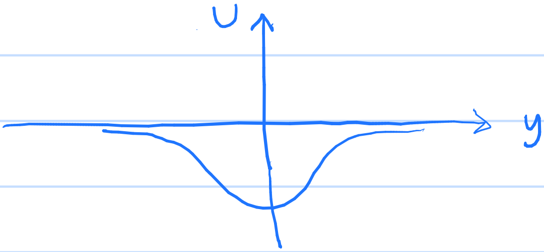
- POTENTIAL ENERGY OF ATOM DUE TO LIGHT



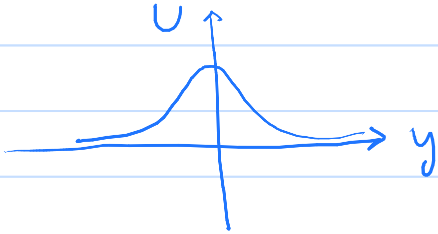


PREVIEW:

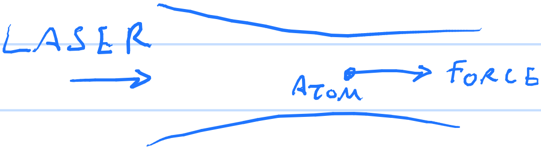
RED-DETUNED: $\omega < \omega_0$ ATTRACTIVE



BLUE-DETUNED: $\omega > \omega_0$ REPULSIVE



2) SCATTERING FORCE - MOMENTUM OF ABSORBED LIGHT



1) DIPOLE FORCE

a) INDUCED DIPOLE IN STATIC (DC) E FIELD

i.e ATOM

- HERE'S A CUTE ARGUMENT THAT GIVES INTUITION AND THE RIGHT ANSWER
- RIGOROUS DERIVATION IN LORENTZ OSCILLATOR NOTES (PDF) SECTION 3.1.2

$\vec{E} = E\hat{x}$ (DC)

$\vec{d} = d_x \hat{x}$; $d_x = -ex = \alpha E_x = \alpha E$

$\hookleftarrow$ REAL @ $\omega = 0$

INCREASE E (SLOWLY) by dE. WORK-ENERGY THEOREM:

$$dU = -F\,dx = -(-eE)\,dx = -E\,d(-ex)$$
$$= -E\,d(\alpha E) = -\alpha E\,dE$$

$$U = -\alpha \int_0^E E'\,dE' = \boxed{-\tfrac{1}{2}\alpha E^2} = -\tfrac{1}{2}\underbrace{\alpha E}_{d_x} E$$

$$= -\tfrac{1}{2} d_x E_x$$

ARB. DIRECTION: $U = -\frac{1}{2}\vec{d}\cdot\vec{E}$ (INDUCED, DC)

$= -\frac{1}{2}\alpha \vec{E}^2$

FORCE: $\vec{F} = -\nabla U = \frac{1}{2}\alpha \nabla(\vec{E}^2)$

- GRAD IN $|\vec{E}| \Rightarrow$ FORCE

b) STATIC (PERMANENT) DIPOLE IN STATIC $\vec{E}$ FIELD:

SKIP

$U = -\vec{d}\cdot\vec{E}$

DERIVATION: $\vec{E} = -\nabla\phi \Rightarrow \phi = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{r}\,' = -\vec{E}\cdot\vec{r}$

$\uparrow$ const

$$U = -\alpha \int_{0\,1}^{E} E'\,dE'_{2} = -\tfrac{1}{2}\alpha E^2$$

+e $\longrightarrow$ −e

FIXED DISTANCE

$$U = \sum_i q_i \phi(\vec{r}_i) = -e(-\vec{E}\cdot\vec{r}_2) + e(-\vec{E}\cdot\vec{r}_1)$$
$$= -(\underbrace{-e\vec{r}_2 + e\vec{r}_1}_{\vec{d}})\cdot\vec{E} = -\vec{d}\cdot\vec{E}$$

- COULD WRITE SAME FORMULAS AS ABOVE, BUT DOESN'T GIVE THE ACTUAL ENERGY STORED IN THE SYSTEM
- HAVE TO ACCOUNT FOR THE FACT THAT THE DIPOLE IS INDUCED

SKIP

b) INDUCED DIPOLE, AC $\vec{E}$ FIELD

TIME-AVG POTENTIAL:

$$U = \left\langle -\tfrac{1}{2}\vec{d}\cdot\vec{E} \right\rangle_t$$

- DERIVATION: LORENTZ NOTES 3.2

CONSIDER $\vec{E} = E_0 \hat{x} \cos(\omega t)$

$$x(t) = \mathcal{U}\cos(\omega t) - \mathcal{V}\sin(\omega t) = \text{Re}\Big[\overbrace{(\mathcal{U}-i\mathcal{V})}^{\tilde{x}}e^{-i\omega t}\Big]$$

$$d_x(t) = -e\,x(t)$$

FIND U:

$$U = \left\langle -\tfrac{1}{2} d_x E_x \right\rangle_t = \left\langle -\tfrac{1}{2}(-ex)E_0\cos(\omega t)\right\rangle_t$$

$$= \tfrac{1}{2}eE_0\left\langle \mathcal{U}\cancelto{1/2}{\cos^2(\omega t)} - \mathcal{V}\underbrace{\cancelto{0}{\sin(\omega t)\cos(\omega t)}}\right\rangle_t$$

$$= \tfrac{1}{4}eE_0\,\mathcal{U}$$

RELATE $\mathcal{U}$ to $\alpha(\omega)$: $-e\tilde{x} = -e(\mathcal{U}-i\mathcal{V}) = \tilde{d}_x = \alpha(\omega)E_0$

$$e\mathcal{U} = -\text{Re}[\tilde{d}_x] = -\text{Re}[\alpha(\omega)]E_0$$

$$e\mathcal{V} = \text{Im}[\tilde{d}_x] = E_0\,\text{Im}[\alpha(\omega)]$$

} SAVE FOR LATER

$$U = \tfrac{1}{4}E_0(e\mathcal{U}) = \tfrac{1}{4}E_0\left(-\text{Re}[\alpha(\omega)]E_0\right)$$

$$= -\tfrac{1}{4}\text{Re}[\alpha(\omega)]E_0^2 \qquad \propto I$$

INTENSITY (VACUUM): $I = \frac{1}{2}\varepsilon_0 c E_0^2$

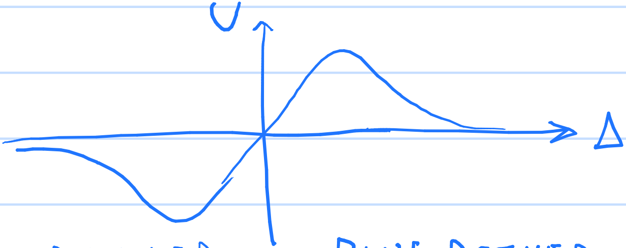
$$\boxed{U = -\frac{I}{2\varepsilon_0 c}\text{Re}[\alpha(\omega)]}$$

NEAR-RESONANCE APPROX: $\alpha(\omega) \approx \frac{-e^2}{2m\omega_0} \frac{\Delta - i\gamma/2}{\Delta^2 + (\gamma/2)^2}$

$\Delta = \omega - \omega_0$

$$\mathrm{Re}[\alpha(\Delta)] = \frac{-e^2}{2m\omega_0} \frac{\Delta}{\Delta^2 + (\gamma/2)^2}$$

$$U = -\frac{I}{2\varepsilon_0 c} \mathrm{Re}[\alpha(\omega)] \approx \frac{I}{2\varepsilon_0 c} \frac{e^2}{2m\omega_0} \frac{\Delta}{\Delta^2 + (\gamma/2)^2}$$



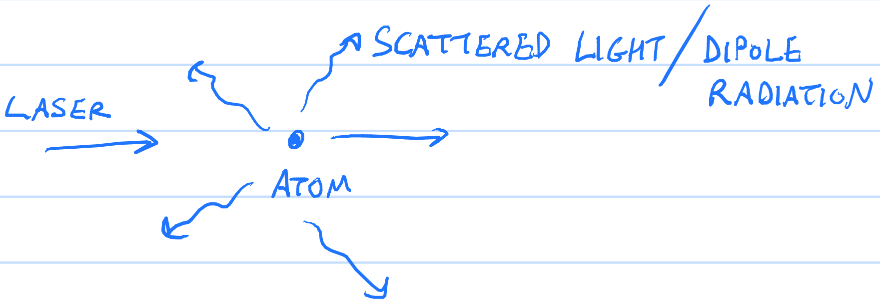
RED-DETUNED

ATTRACTIVE

BLUE-DETUNED

REPULSIVE

2) SCATTERING FORCE



MOMENTUM OF LIGHT

- CLASSICAL: $\text{MOMENTUM DENSITY} = \dfrac{\text{ENERGY DENSITY}}{c}$
- QUANTUM + RELATIVITY: $E = \sqrt{(pc)^2 + (mc^2)^2}$

photon: $m = 0$, $E_{photon} = pc$

$\rightarrow p = E/c$

$$\Rightarrow \text{FORCE} = \frac{\text{MOMENTUM}}{\text{TIME}} = \frac{\text{ENERGY ABSORBED}/c}{\text{TIME}}$$

$$= \frac{\text{POWER ABSORBED}}{c}$$

$$\text{Power} = \text{FORCE} \times \text{VELOCITY}$$

ASSUME $\dot{x}_n = 0$ FOR SIMPLICITY

$$P_{abs} = \left\langle F_e \dot{x}_e + \cancelto{\approx 0}{F_n x_n} \right\rangle_t$$

$$F_e = -eE_x = -eE_0 \cos(\omega t)$$

$$\dot{x}_e = \frac{d}{dt} x_e = \frac{d}{dt}\left(U\cos(\omega t) - V\sin(\omega t)\right)$$

$$= -\omega U \sin(\omega t) - \omega V \cos(\omega t)$$

$$P_{abs} = eE_0\omega\left\langle U\,\underline{\cancelto{0}{\cos(\omega t)\sin(\omega t)}} + V\cancelto{1/2}{\cos^2(\omega t)}\right\rangle_t$$

$$= \frac{1}{2}eE_0\omega V$$

$$F_{scatt} = \frac{P_{abs}}{c} = \frac{e\omega}{2c}E_0 V$$

RELATE V TO $\alpha(\omega)$:

$$-e\tilde{x} = -e(U - iV) = \tilde{d}_x = \alpha(\omega)E_0$$

$$eV = \text{Im}[\tilde{d}_x] = E_0\,\text{Im}[\alpha(\omega)]$$

PHY 446 SPRING 2020
LECTURE 5

- REVIEW OF HYDROGEN ATOM & PAULI PRINCIPLE
- IDENTICAL PARTICLES

WARM-UP / REVIEW

WRITE GROUND-STATE ELECTRON CONFIGURATION OF:

Z	symbol	config.
1	H	$1s$
2	He	$1s^2$
3	Li	$1s^2 2s$
10	Ne	$1s^2 2s^2 2p^6$
11	Na	$1s^2 2s^2 2p^6 3s$

RULES:

	ℓ
s	0
p	1
d	2
f	3

$\ell < n$

EACH ELECTRON HAS n, ℓ, m_ℓ, m_s

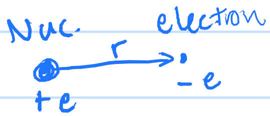
PAULI PRINCIPLE : NO TWO ELECTRONS CAN HAVE THE SAME QUANTUM NUMBERS

FOR $Z \leq 56$ (Ba) (GROUND STATE)

$$E_{ns} < \underbrace{E_{(n-1)d} < E_{np}}_{\text{If exist}} < E_{(n+1)s}$$

- AFTER THAT, NEED f ORBITALS, TOO

HYDROGEN $(Z=1)$



ELECTRON WAVEFUNCTIONS

$$\psi(\vec{r}) = \psi_{n\ell m}(r,\theta,\phi)$$

$$= R_{n\ell}(r)\, Y_{\ell m}(\theta,\phi)$$

GRIFFITHS: Y_ℓ^m , FOOT: $Y_{\ell m}$

$Y_{\ell m}$ ARE ANGULAR MOMENTUM EIGENFUNCTIONS

$$\hat{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$\hat{L}_z Y_{\ell m} = \hbar m\, Y_{\ell m} \qquad \left(\hat{L}_z = -i\hbar \frac{\partial}{\partial\phi}\right)$$

$R_{n\ell}$ SATISFIES THE RADIAL EQN

LET $u = r \cdot R(r)$

$$-\frac{\hbar^2}{2m_e}\frac{d^2u}{dr^2} + \underbrace{\left(\frac{\hbar^2}{2m_e}\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{4\pi\varepsilon_0 r}\right)}_{V_{eff}} u = Eu$$

SOLUTIONS:

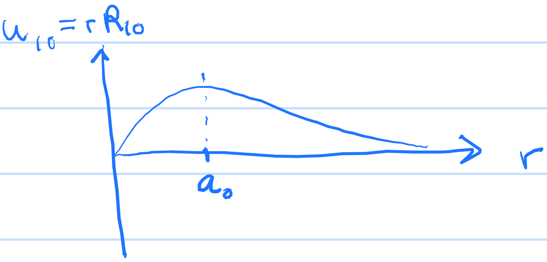
$$E_{n\ell} = E_n = \frac{-m_e\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2}{2\hbar^2}\,\frac{1}{n^2} = \frac{-13.6\,eV}{n^2}$$

$n > \ell$

$n=1, \ell=0$ (1s) GND STATE

$R_{10}(r) \propto e^{-r/a_0}$

a_0 = BOHR RADIUS = $\frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.529\times10^{-10}$ M



(IT FOLLOWS A SCHRÖD-LIKE EQN.)

ASIDE – WHY LOOK AT $u_{n\ell}$? ANOTHER REASON:

RADIAL PROB. DENSITY

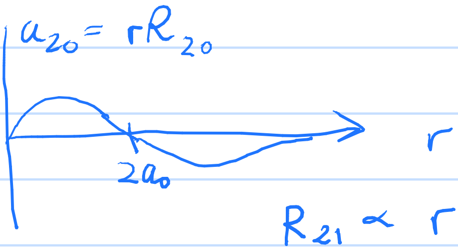
$$P(r)\,dr = \int_0^\pi d\theta \int_0^{2\pi} d\phi\, |R_{n\ell}(r)\, Y_{\ell m}(\theta,\phi)|^2\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= r^2 R_{n\ell}^2(r)\,dr = (r R_{n\ell}(r))^2 = u_{n\ell}^2(r)\,dr$$

$\rightarrow P(r) = |u_{n\ell}(r)|^2$

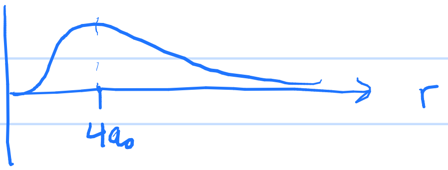
FIRST EXCITED STATES $n=2$

$\ell=0$: $R_{20}(r) \propto (2-r/a_0)\,e^{-r/(2a_0)}$



$R_{21} \propto r\, e^{-r/2a_0}$

$\ell=1$:



- "ACCIDENTAL" DEGENERACY IN ℓ
  - FOR ARB $V(r)$, $E_{n\ell}$ DEPENDS ON ℓ

RADIAL NODES: ZEROS OF $u(r)$ FOR $\underline{r>0}$

- $r=0$ DOESN'T COUNT

Radial nodes $\nu = n-\ell-1$

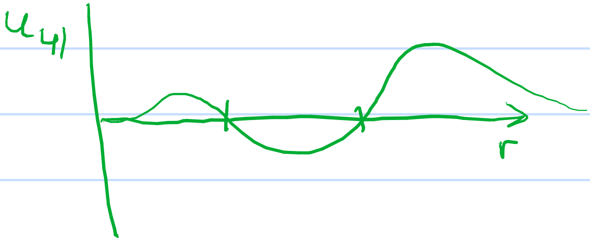
$\rightarrow n = \nu+\ell+1$

n	ℓ	ν	
1	0	0	✓
2	0	1	✓
2	1	0	✓

E.G. SKETCH $u(r) = rR(r)$ FOR 4p WVFN

$n=4, \ \ell=1$

$\nu = n-\ell-1 = 4-1-1 = \boxed{2}$



MULTIELECTRON ATOMS

- HOW TO: WAVEFUNCTIONS FOR MULTIPLE ELECTRONS
- ORIGIN OF PAULI PRINCIPLE

CONSIDER TWO IDENTICAL PARTICLES (1 & 2)

i.e. TWO ELECTRONS

WITH SAME SPIN PROJECTION: $m_{s1} = m_{s2}$

i.e. ↑↑

$$\overset{\uparrow}{\phi}_1 \quad \overset{\uparrow}{\phi}_2$$

TWO-PARTICLE WAVEFUNCTION $\psi(\vec{r}_1, \vec{r}_2)$

MEANING: JOINT PROB. DENSITY IS

$$P(\vec{r}_1, \vec{r}_2) = |\psi(\vec{r}_1, \vec{r}_2)|^2$$

PROB OF $\vec{r}_1$ WITHIN d^3r_1 OF $\vec{r}_1{}'$

& $\vec{r}_2$ WITHIN d^3r_2 OF $\vec{r}_2{}'$

$$= |\psi(\vec{r}_1, \vec{r}_2)|^2 \, d^3r_1 \, d^3r_2$$

EXCHANGE SYMMETRY

- PARTICLES ARE INDISTINGUISHABLE, SO LABELS "1" AND "2" HAVE NO REAL MEANING

→ $P(r_1, r_2) = P(r_2, r_1)$

→ $|\psi(r_1, r_2)| = |\psi(r_2, r_1)|$

→ $\psi(r_1, r_2) = e^{i\theta} \psi(r_2, r_1)$

QUANTUM STATISTICS

BOSONS: $e^{i\theta} = 1$

$\Psi(r_1, r_2) = \Psi(r_2, r_1)$ (EQUAL SPIN)

- E.G. PIONS, PHOTONS, W & Z BOSONS

FERMIONS: $e^{i\theta} = -1$

$\Psi(r_2, r_1) = -\Psi(r_1, r_2)$ (EQUAL SPIN)

- E.G. ELECTRONS, PROTON, NEUTRONS

SPIN-STATISTICS CONNECTION

INTEGER SPIN (EG 0, 1, 2 ...) → BOSON

HALF-INTEGER SPIN (EG. 1/2, 3/2, ...) → FERMION

ELECTRON: SPIN $\frac{1}{2}$ → FERMION

COMPOSITE PARTICLES: EVEN # of FERMI → BOSE

ODD # of FERMI → FERMI

E.G. ^1H ATOM → BOSON

^4He ATOM: $nnppee$ → BOSON

^6Li ATOM: $n^3p^3e^3$ → FERMION

— END of LECTURE

PRODUCT WAVEFUNCTIONS

- FOR TWO INDISTINGUISHABLE PARTICLES

$$\Psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2)$$

OR $\psi_b(r_1)\psi_a(\vec{r}_2)$ ← SAME ENERGY

ANY LINEAR COMB. HAS SAME ENERGY

SYMMETRIZE: $\Psi(\vec{r}_1, \vec{r}_2) = \pm\Psi(\vec{r}_1, \vec{r}_2)$

$$\rightarrow \Psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]$$

FOR ELECTRONS (FERMIONS) WITH SAME SPIN, (↑↑)

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left[\psi_a(r_1)\psi_b(r_2) - \psi_b(r_1)\psi_a(r_2)\right]$$

PAULI PRINCIPLE: CAN'T HAVE $\psi_a = \psi_b$,

B/C THEN

$$\Psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_a(r_1)\psi_a(r_2) - \psi_a(r_1)\psi_a(r_2)\right)$$
$$= 0$$

WAVEFUNCTION SYMMETRY

→ TWO ELECTRONS CAN'T BE IN SAME QUANTUM STATE ie $|\psi_a, \uparrow\rangle$

SEPARATION OF VARIABLES

FOR NON-INTERACTING PARTICLES, POTENTIAL ENERGY

$$U(\vec{r}_1, \vec{r}_2) = V(\vec{r}_1) + V(\vec{r}_2)$$

→ TISE SOLVED BY SEP. OF VARS

HELIUM GROUND STATE

CONFIGURATION: $1s^2$

PAULI: THE ELECTRONS HAVE OPPOSITE SPIN

PHY 446 SPRING 2020
LECTURE 6

SURVEY OF ATOMIC STRUCTURE: PART 2

- ORIGIN OF PAULI PRINCIPLE
- LS COUPLING
- FINE & HYPERFINE STRUCTURE

MULTI-ELECTRON WAVEFUNCTIONS

FOR TWO ELECTRONS W/ SAME SPIN (i.e. ↑↑)

- INDISTINGUISHABLE FERMIONS

$\psi(\vec{r}_1, \vec{r}_2) = -\psi(\vec{r}_2, \vec{r}_1)$

PRODUCT WAVEFUNCTIONS ("INDEPENDENT ELECTRON APPROX.")

- USE PRODUCT OF SINGLE-PARTICLE WAVEFUNCTIONS

$$\psi_a(r_1)\psi_b(r_2) \quad \text{OR} \quad \psi_b(r_1)\psi_a(r_2)$$

i.e. $\psi_a = \psi_{n\ell m}$; $\psi_b = \psi_{n'\ell' m'}$

- MUST BE ANTI-SYMMETRIC:

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left[\psi_a(r_1)\psi_b(r_2) - \psi_b(r_1)\psi_a(r_2)\right]$$

– CHECK!

PAULI PRINCIPLE: CAN'T HAVE $\psi_a = \psi_b$,

B/C THEN

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_a(r_1)\psi_a(r_2) - \psi_a(r_1)\psi_a(r_2)\right)$$
$$= 0$$

→ TWO ELECTRONS CAN'T BE IN SAME QUANTUM STATE ie $|n\ell m, \uparrow\rangle$

E.X. He GROUND STATE $1s^2$

IF SPIN ↑↑ THEN e^-'S INDISTINGUISHABLE

→ CAN'T BOTH HAVE $n=1, \ell=0, m_\ell=0$

SO MUST HAVE OPPOSITE SPIN

⇒ TOTAL SPIN MUST BE $S=0$

ALSO, TOTAL ORBITAL $L=0$

LS COUPLING

CONSIDER ATOM WITH N ELECTRONS

TOTAL ORBITAL ANGULAR MOMENTUM OF ELECTRONS:

$$\vec{L} \equiv \vec{\ell}_1 + \vec{\ell}_2 + \cdots + \vec{\ell}_N = \sum_{i=1}^{N} \vec{\ell}_i$$

TOTAL SPIN:

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \cdots + \vec{S}_N = \sum_{i=1}^{N} \vec{S}_i$$

MOST ENERGY LEVELS IN MOST ATOMS ARE APPROX. EIG. STATES OF:

$\vec{L}^2$ AND $\vec{S}^2$

WHY? BECAUSE: $[\hat{H}, \hat{L}_\alpha] \approx 0$

HAMILTONIAN OF ELECTRONS IN ATOM

$\alpha = x, y, z$ COMPONENTS

AND: $[\hat{H}, \hat{S}_\alpha] \approx 0$

- OPERATORS THAT COMMUTE ARE "COMPATIBLE": CAN KNOW SIMULTANEOUSLY

- RECALL: $[\hat{L}^2, \hat{L}_z] = 0$

  $[\hat{S}^2, \hat{S}_z] = 0$

  $[\hat{S}_\alpha, \hat{L}_\beta] = 0$ for $\alpha, \beta = x, y, z$

BUT: $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ etc.

$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$ etc

$$\begin{tikzcd} & x \arrow[dr] & \\ z \arrow[ur] & & y \arrow[ll] \end{tikzcd}$$

SO CAN ONLY KNOW ONE COMPONENT of $\vec{L}$ (i.e. L_z) AND ONE of $\vec{S}$ (i.e. S_z)

EIGENVALUES OF $\hat{L}^2$ ARE $\hbar^2 L(L+1)$

OF $\hat{S}^2$ ARE $\hbar^2 S(S+1)$

- POSSIBLE VALUES of L & S DEPEND ON THE CONFIGURATION, i.e. $1s^2 2s^2 2p \ldots$

  — POSSIBLE TO PREDICT, BUT WON'T COVER THAT TODAY

RUSSELL-SAUNDERS (SPECTROSCOPIC) NOTATION:

LABEL ATOMIC STATES USING

^{2S+1}L "TERM SYMBOL"

$\uparrow$ USE LETTER HERE

E.x.

1) H GROUND STATE: $1s \rightarrow S = \frac{1}{2}$, $L = 0$

   $2S+1 = 2$; $L \rightarrow$ "S"

   ^2S

2) Li GROUND STATE: $1s^2 2s \rightarrow S = \frac{1}{2}$, $L = 0$

   ^2S

3) C ($Z=6$) GND STATE: $1s^2 2s^2 2p^2$: $S = 1$, $L = 1$

   ^3P

   EVEN # of e^- $\Rightarrow$ S INT.

4) C: $1s^2 2s^2 2p^2$: $S = 0$, $L = 2$ ^1D (981 nm)

FINE STRUCTURE

SPECIAL RELATIVITY EFFECT

IN ELECTRON FRAME, $\vec{E}$ FIELD OF NUCLEUS INDUCES A $\vec{B}$ FIELD:

$$\vec{B} = -\frac{1}{c^2}\vec{v}\times\vec{E} \qquad \text{(RELATIVISTIC TRANSFORMATION)}$$

$$= -\frac{1}{c^2}\vec{v}\times\left(\frac{e}{4\pi\varepsilon_0 r^2}\,\frac{\vec{r}}{r}\right)$$

$$\propto \vec{r}\times(m\vec{v}) = \vec{r}\times\vec{p} = \vec{\ell}$$

ELECTRON SPIN INTERACTS WITH $\vec{B}$

$$H_{so} = -\vec{\mu}_s\cdot\vec{B} \propto \vec{s}\cdot\vec{\ell}$$

$$H_{so} \equiv \beta\,\vec{s}\cdot\vec{\ell}$$

FOR MULTIPLE ELECTRONS,

$$H_{so} \approx \beta\,\vec{S}\cdot\vec{L}$$

TOTAL SPIN

FIND EIGENSTATES OF H_{so}:

DEFINE TOTAL ELECTRON ANGULAR MOMENTUM

$$\vec{J} = \vec{L}+\vec{S}$$

TRICK: $\vec{J}^2 = \vec{J}\cdot\vec{J} = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) = \vec{L}^2 + 2\vec{S}\cdot\vec{L} + \vec{S}^2$

$$\vec{S}\cdot\vec{L} = \frac{1}{2}\left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right]$$

EIGENVALUES:

$\hat{J}^2$: $\hbar^2 J(J+1)$

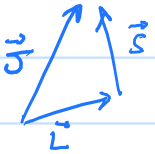
$\vec{S}\cdot\vec{L}$: $\frac{1}{2}\hbar^2\left[J(J+1)-L(L+1)-S(S+1)\right]$

$$E_{so} = \frac{\beta}{2}\hbar^2\left[J(J+1)-L(L+1)-S(S+1)\right]$$

ANGULAR MOMENTUM ADDITION RULE:

$$J = |L-S|,\ |L-S|+1,\ \ldots,\ L+S$$

i.e.



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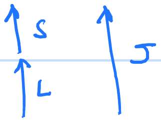

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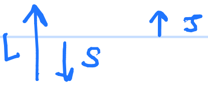


EX. Na EXCITED STATES (Z=11)

GND $1s^2 2s^2 2p^6 3s$

EXC. $\underbrace{1s^2 2s^2 2p^6}_{L=0,\ S=0} 3p$ ← VALENCE ELECTRON HAS ALL THE ANGULAR MOMENTUM

FOR FILLED SHELLS

a) TOTAL L & S IN EXC. STATE

$L=1,\quad S=\frac{1}{2}$

b) TERM SYMBOL: ${}^2P$

c) POSSIBLE J VALUES: $J = 1-\frac{1}{2},\ 1+\frac{1}{2} = \boxed{\frac{1}{2},\ \frac{3}{2}}$

${}^{2S+1}L_J = {}^2P_{\frac{1}{2}},\quad {}^2P_{3/2}$ ← "LEVEL"

Na GND STATE $J = |0-\frac{1}{2}|, 0+\frac{1}{2} = \frac{1}{2}$

TERM & LEVEL SYMBOL: $^2S_{1/2}$

Na ENERGY LEVELS (FINE STRUCTURE) – FIRST FEW

	"GROSS" STRUCTURE		FINE STRUCTURE		
EXC.	3p	^2P	3p	$J=3/2$	$^2P_{3/2}$
				$J=1/2$	$^2P_{1/2}$
GND	3s	^2S	3s	$J=\frac{1}{2}$	$^2S_{1/2}$

ENERGY SCALES – NIST DATABASE

$$cm^{-1} = \frac{1}{\lambda} = \frac{E}{hc}$$

GND: $\frac{1}{hc}E(^2S_{1/2}) \equiv 0.0 \text{ cm}^{-1}$

EXC: $\frac{1}{hc}E(^2P_{1/2}) = 16{,}956.2 \text{ cm}^{-1}$

$\Rightarrow \lambda = 589.76 \text{ nm}$ "D1 LINE" (YELLOW)

$\frac{1}{hc}E(^2P_{3/2}) = 16{,}973.4 \text{ cm}^{-1}$

$\Rightarrow \lambda = 589.16 \text{nm}$ "D2 LINE"

SPLITTING:

$E(^2P_{3/2}) - E(^2P_{1/2}) = 17.2 \text{ cm}^{-1}$

$\Rightarrow \lambda = 0.581 \underline{\text{mm}}$

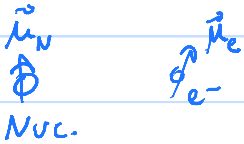
$f = \frac{c}{\lambda} = 516 \text{ GHz}$

(FAR INFRARED/ THz BAND)

HYPERFINE STRUCTURE

- FINE STRUCTURE SUFFICES FOR MANY PURPOSES BUT SOMETIMES NEED MORE PRECISION

EFFECT OF NUCLEAR SPIN MAGNETIC MOMENT



(~ TWO MAGNETS)

e^- PRODUCES $\vec{B}$ FIELD, $\vec{B}_e \propto -\vec{J}$ (APPROX.)

INTERACTS W/ MAGNETIC MOMENT OF NUC.

LET $\vec{I}$ = NUC. SPIN ANG. MOMENTUM

$\vec{\mu}_N \propto \vec{I}$

ENERGY OF $\vec{\mu}_N$ IN $\vec{B}_e$: $-\vec{\mu}_N \cdot \vec{B}_e \propto \vec{I} \cdot \vec{J}$

$$\hat{H}_{HF} \approx A\, \vec{I} \cdot \vec{J}$$

SAME TRICK: $\vec{F} \equiv \vec{I} + \vec{J}$ (TOTAL INTERNAL ANGULAR MOMENTUM OF ATOM)

$$\vec{F}^2 = (\vec{I}+\vec{J}) \cdot (\vec{I}+\vec{J}) = \vec{I}^2 + 2\vec{I}\cdot\vec{J} + \vec{J}^2$$

$$\vec{I} \cdot \vec{J} = \frac{1}{2}\left(\vec{F}^2 - \vec{I}^2 - \vec{J}^2 \right)$$

$\hat{H}_{HF}$ EIGENVALUES:

$$E_{HF} = \frac{A}{2} \hbar^2 \left[F(F+1) - I(I+1) - J(J+1) \right]$$

ALLOWED VALUES OF F:

$F = |I-J|, \dots, I+J$

E.x. ^{23}Na: $I = \frac{3}{2}$

GND STATE: 3s

a) FIND S, L, J & TERM/LEVEL SYMBOL (REVIEW)

$S=\frac{1}{2}, L=0, J=\frac{1}{2}$; $^2S_{1/2}$

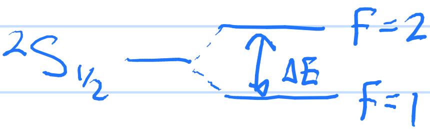
b) FIND ALLOWED F

$F_{min} = |I-S| = \frac{3}{2} - \frac{1}{2} = 1$

$F_{max} = I+S = \frac{3}{2} + \frac{1}{2} = 2$

$\boxed{F = 1, 2}$

DIAGRAM:



$\Delta E = hf$; $f = 1.771$ GHz

PHY 446 SPRING 2020
LECTURE 7

2/10/2020

- REVIE ANG. MOM. ADDITION
- FINE + HYPERFINE STRUCTURE

WARM-UP

EX. Na EXCITED STATES (Z=11)

GND $1s^2 2s^2 2p^6 3s$

EXC. $\underbrace{1s^2 2s^2 2p^6}_{L=0,\ S=0} 3p$ ← VALENCE ELECTRON HAS ALL THE ANGULAR MOMENTUM

FOR FILLED SHELLS

a) TOTAL L + S QUANTUM #'S IN EXC. STATE

$L=1, \quad S=\frac{1}{2}$

b) TERM SYMBOL: ${}^2P$

c) POSSIBLE J VALUES: $J = 1-\frac{1}{2},\ 1+\frac{1}{2} = \boxed{\frac{1}{2},\ \frac{3}{2}}$

$${}^{2S+1}L_J = {}^2P_{\frac{1}{2}},\quad {}^2P_{3/2}$$ ← "LEVEL"

Na GND STATE $J = |0-\frac{1}{2}|, 0+\frac{1}{2} = \frac{1}{2}$

TERM & LEVEL SYMBOL: $^2S_{1/2}$

Na ENERGY LEVELS (FINE STRUCTURE) – FIRST FEW

	"GROSS" STRUCTURE		FINE STRUCTURE		
EXC.	3p ——	^2P	3p ——	$J=3/2$	$^2P_{3/2}$
			——	$J=1/2$	$^2P_{1/2}$
GND	3s ——	^2S	3s ——	$J=\frac{1}{2}$	$^2S_{1/2}$

ENERGY SCALES – NIST DATABASE

$$\text{cm}^{-1} = \frac{1}{\lambda} = \frac{E}{hc}$$

GND: $\frac{1}{hc}E(^2S_{1/2}) \equiv 0.0 \text{ cm}^{-1}$

EXC: $\frac{1}{hc}E(^2P_{1/2}) = 16{,}956.2 \text{ cm}^{-1}$

$\Rightarrow \lambda = 589.76$ nm "D1 LINE" (YELLOW)

$\frac{1}{hc}E(^2P_{3/2}) = 16{,}973.4 \text{ cm}^{-1}$

$\Rightarrow \lambda = 589.16$nm "D2 LINE"

SPLITTING:

$E(^2P_{3/2}) - E(^2P_{1/2}) = 17.2 \text{ cm}^{-1}$

$\Rightarrow \lambda = 0.581 \ \underline{\text{mm}}$

$f = \frac{c}{\lambda} = 516$ GHz

(FAR INFRARED/ THz BAND)

ANGULAR MOMENTUM ADDITION

TWO ANGULAR MOMENTA, i.e. $\vec{L}$ & $\vec{S}$
LET $\vec{J} = \vec{L} + \vec{S}$

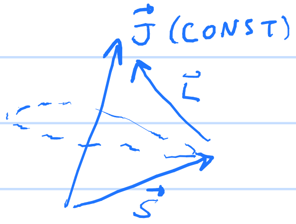
TWO WAYS TO REPRESENT THE QUANTUM STATES

1. USING L_z, S_z "UNCOUPLED BASIS"
2. USING J^2, J_z "COUPLED BASIS"

USE COUPLED BASIS B/C $\vec{L}$ & $\vec{S}$ INTERACT

- $\vec{L}$ & $\vec{S}$ INTERACT ⇒ $\vec{L}, \vec{S}$ NOT CONSTANT
- NO EXTERNAL TORQUE ⇒ $\vec{J}$ CONSTANT

CLASSICAL



- LENGTHS $|\vec{L}|, |\vec{S}|$ ARE CONST. BUT DIRECTIONS CHANGE

DECOUPLED BASIS

QUANTUM STATES: $|L\,S\,M_L M_s\rangle \equiv |M_\ell M_s\rangle$

$$\vec{L}^2 |M_L M_s\rangle = \hbar^2 L(L+1)|M_L M_s\rangle$$

$$\hat{S}^2 |M_L M_s\rangle = \hbar^2 S(S+1)|M_L M_s\rangle$$

$$\hat{L}_z |M_L M_s\rangle = \hbar M_L |M_L M_s\rangle$$

$$\hat{S}_z |M_L M_s\rangle = \hbar M_s |M_L M_s\rangle$$

COUPLED BASIS

(TOTAL $\vec{J} = \vec{L} + \vec{S}$)

$\vec{J}$ OBEYS ANG. MOM. COMMUTATION RULES

$$[J_x, J_y] = i\hbar J_z \quad etc$$

$$\Rightarrow [\vec{J}^2, J_z] = 0$$

→ FIND SIMULTANEOUS EIG. STATES

INTRODUCE QUANTUM NUMBERS J, M_J

COUPLED BASIS: $|L, S, JM_J\rangle = |JM_J\rangle$

EIG. VALUES OF J^2, J_z :

$$\vec{J}^2 |JM_J\rangle = \hbar^2 J(J+1) |JM_J\rangle$$

$$J_z |JM_J\rangle = \hbar M_J |JM_J\rangle$$

WHERE $J = |L-S|, \ldots, L+S$

$M_J = -J, \ldots, J$

COUNT: [let $j_2 = MAX(L,S)$, $j_1 = MIN(L,S)$]

BASIS	# STATES
$\lvert M_L M_S\rangle$	$(2L+1)(2S+1)$
$\lvert JM_J\rangle$	$\left.\begin{array}{l}[2(j_2-j_1)+1] + 2(j_2-j_1+1)+1 \\ \quad \ldots + 2(j_1+j_2)+1\end{array}\right\}$ ARITHMETIC SERIES

$$= \underbrace{\left[\frac{2(2j_2)+2}{2}\right]}_{MEAN} \underbrace{\left(j_1+j_2-(j_2-j_1)+1\right)}_{NUMBER}$$

$$= (2j_2+1)(2j_1+1) = (2L+1)(2S+1)$$

CHANGE OF BASIS FORMULA:

FOR ANY BASIS $\{|n\rangle\}$

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle$$

RELATE COUPLED & DECOUPLED BASES

$$|JM_J\rangle = \sum_{m_L m_s} |m_L m_s\rangle \underbrace{\langle m_L m_s | J M_J\rangle}_{\text{CLEBSCH-GORDAN COEF. (REAL-VALUED)}}$$

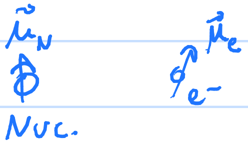
LIKEWISE,

$$|M_L M_s\rangle = \sum_{J, M_J} |J M_J\rangle \underbrace{\langle J M_J | M_L M_S\rangle}_{\text{SAME CG COEFFICIENTS}}$$

HYPERFINE STRUCTURE

- FINE STRUCTURE SUFFICES FOR MANY PURPOSES BUT SOMETIMES NEED MORE PRECISION

EFFECT OF NUCLEAR SPIN MAGNETIC MOMENT



(~ TWO MAGNETS)

e^- PRODUCES $\vec{B}$ FIELD, $\vec{B}_e \propto -\vec{J}$ (APPROX.)

INTERACTS W/ MAGNETIC MOMENT OF NUC.

LET $\vec{I}$ = NUC. SPIN ANG. MOMENTUM

$\vec{\mu}_N \propto \vec{I}$

ENERGY OF $\vec{\mu}_N$ IN $\vec{B}_e$: $-\vec{\mu}_N \cdot \vec{B}_e \propto \vec{I} \cdot \vec{J}$

$$\hat{H}_{HF} \approx A\, \vec{I} \cdot \vec{J} / \hbar^2$$

SAME TRICK: $\vec{F} \equiv \vec{I} + \vec{J}$ (TOTAL INTERNAL ANGULAR MOMENTUM OF ATOM)

$$\vec{F}^2 = (\vec{I} + \vec{J}) \cdot (\vec{I} + \vec{J}) = \vec{I}^2 + 2\vec{I}\cdot\vec{J} + \vec{J}^2$$

$$\vec{I} \cdot \vec{J} = \frac{1}{2}\left(\vec{F}^2 - \vec{I}^2 - \vec{J}^2 \right)$$

$\hat{H}_{HF}$ EIGENVALUES:

$$E_{HF} = \frac{A}{2}\left[F(F+1) - I(I+1) - J(J+1) \right]$$

ALLOWED VALUES OF F:

$F = |I-J|, \ldots, I+J$

E.X. ^{23}Na: $I = \frac{3}{2}$

GND STATE: 3s

a) FIND S, L, J & TERM/LEVEL SYMBOL (REVIEW)

$S = \frac{1}{2},\ L = 0,\ J = \frac{1}{2}\ ;\ {}^2S_{1/2}$

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$F_{min} = |I-S| = \frac{3}{2} - \frac{1}{2} = 1$

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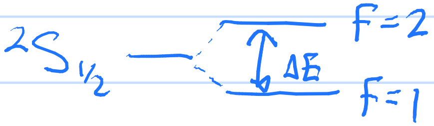


$\Delta E = hf\ ;\ f = 1.771\ \text{GHz}$

ATOMIC HAMILTONIAN (NO APPLIED FIELDS)

$$H_0 = \underbrace{H_{NR}}_{\text{NON-RELATIVISTIC}} + \underbrace{H_{FS}}_{\text{FINE-STRUCT.}} + \underbrace{H_{HFS}}_{\text{HYPERFINE STRUCT.}}$$

$$H_{NR} = \sum_{i=1}^{N} \left(\underbrace{\frac{p_i^2}{2m_e}}_{\text{KINETIC}} - \underbrace{\frac{Ze^2}{4\pi\varepsilon_0 r_i}}_{\substack{\text{INTERACTION} \\ \text{W/ NUCLEAR} \\ \text{CHARGE} \\ +Ze}} \right) + \underbrace{\sum_{i<j} \frac{e^2}{4\pi\varepsilon_0 r_{ij}}}_{\substack{\text{ELECTRON-} \\ \text{ELECTRON} \\ \text{INTERACTION}}}$$

H_{NR} COMMUTES W/ $\vec{L}$ & $\vec{S}$

$\Rightarrow$ LS COUPLING

$H_{FS} \approx \frac{\beta}{\hbar^2} \vec{L} \cdot \vec{S}$ COMMUTES W/ $\vec{J}^2$

$\Rightarrow$ SPLIT ACCORDING TO J

$H_{HFS} \approx \frac{A}{\hbar^2} \vec{I} \cdot \vec{J}$ COMMUTES W/ $\vec{F}^2$

$\Rightarrow$ SPLIT ACCORDING TO F

PHY 446 SPRING 2020
LECTURE 8

2/12/2020

- WED FEB 19: SOMMER AWAY / QUIZ / HW2 → MON FEB 24 / BIAGGIO
- TODAY: OPTICAL TRANSITIONS (OVERVIEW)
- SELECTION RULES, 2-LEVEL ATOM

REVIEW: DEFINE THE ANGULAR MOMENTUM VARIABLES

a)	L	ELECTRON ORBITAL ANG. MOMENTUM
b)	S	ELECTRON SPIN
c)	J	$\vec{L}+\vec{S}=\vec{J}$
d)	I	NUCLEAR SPIN
e)	F	TOTAL INTERNAL ANG. MOMENTUM $\vec{I}+\vec{J}$

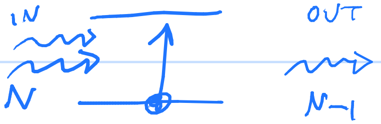
ATOM-LIGHT INTERACTION

PREVIOUS: ATOM ≃ CLASSICAL HARMONIC OSCILLATOR

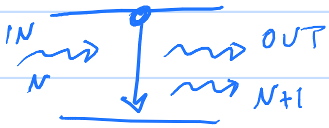
NOW: ATOM = QUANTUM SYSTEM

3 BASIC PROCESSES

1. ABSORPTION



2. STIMULATED EMISSION
   - A NEW, IDENTICAL PHOTON APPEARS

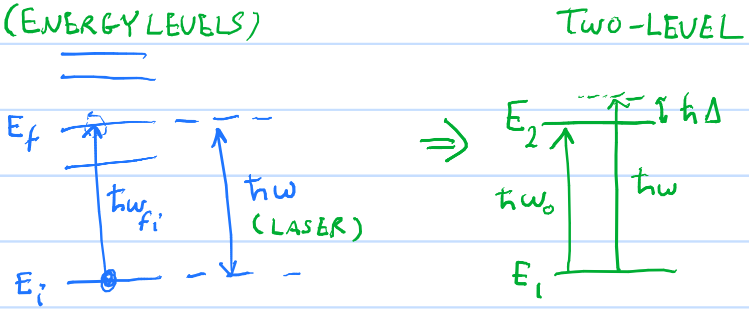


TODAY

3. SPONTANEOUS EMISSION

ABSORPTION

MULTILEVEL ATOM IN EM FIELD (LASER) AT FREQ ω



ATOM EXCITED TO RESONANT OR NEAR-RESONANT LEVEL:

$$\hbar\omega \approx E_f - E_i \quad \text{(ENERGY CONSERVATION)}$$

↳ APPROX. B/C OF TIME-ENERGY UNCERTAINTY

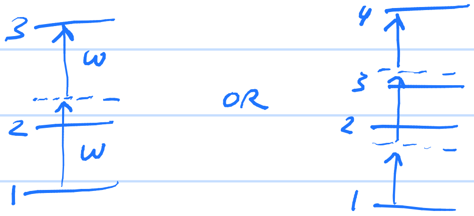
- FINITE PULSE; FINITE EXICTED STATE LIFETIME

CAN NEGLECT OTHER LEVELS

→ TWO-LEVEL MODEL

NOT INCLUDED IN TWO-LEVEL MODEL:

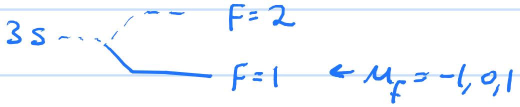
MULTIPHOTON TRANSITIONS



- MULTIPHOTON RESONANCE: NEED 3 OR MORE LEVELS

DEGENERACY IN TWO-LEVEL MODEL

i.e. ^{23}Na GND STATE MANIFOLD



F=1 STATES: $|3s, {}^2S_{1/2}, F=1, M_F\rangle$

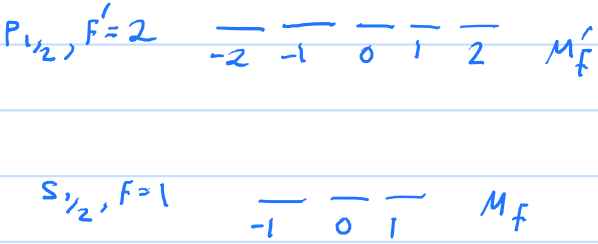
$M_F = -1, 0, 1$ (THREE-FOLD DEGENERATE)

EXCITED STATE ALSO DEGENERATE

i.e. $|3p, {}^2P_{1/2}, F'=2, M_f'\rangle$

FOR $M_f' = -2, -1, 0, 1, 2$

TWO LEVELS, BUT MANY STATES



REDUCTION TO TWO STATES

- ASSUME ATOM STARTS IN A SPECIFIC STATE

e.g. $|\psi(t=0)\rangle = |1\rangle = |3s, {}^2S_{1/2}, F=1, M_F=1\rangle$

FINAL STATE:

- PHOTON CARRIES ANGULAR MOMENTUM $\vec{j}$ ← A.K.A. $\vec{j}_\gamma$

WITH $j=1$ ("SPIN 1")

ATOM ABSORBS PHOTON, GAINS ANGULAR MOMENTUM

$$\vec{F}' = \vec{F} + \vec{j}$$

- ELECTRIC FIELD AT ATOM: $\vec{E}(t) = E_0 \, Re[\hat{\varepsilon} e^{-i\omega t}]$

POLARIZATION $\hat{\varepsilon}$	PHOTON m_j	ΔM_F	SYMBOL
$\hat{\varepsilon}_+ = \frac{-1}{\sqrt{2}}(\hat{x} + i\hat{y})$	1	1	σ^+
$\hat{\varepsilon}_0 = \hat{z}$	0	0	π
$\hat{\varepsilon}_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$	-1	-1	σ^-

"SPHERICAL BASIS"

SELECTION RULES

- ALLOWED $F' = |F-j|, \ldots, F+j$

$= |F-1|, \ldots, F+1$

CASE 1) $F \geq 1$: $F' = F-1, F, F+1$

i.e. $\Delta F = 0, \pm 1$

CASE 2) $F=0$: $F'=1$

i.e. $F=0 \rightarrow F'=0$ FORBIDDEN

- ALLOWED $M_F' = M_F + m_j \Rightarrow \Delta M_F = 0, \pm 1$

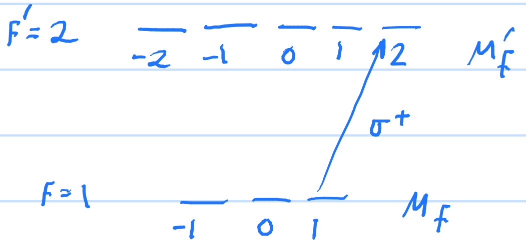
⇒ ONLY ONE ALLOWED FINAL STATE

E.G. FOR σ^+ TRANSITION, $M_F' = M_F + 1$

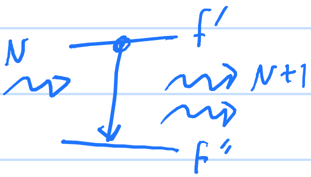
CONSIDER $F=1 \rightarrow F'=2$ TRANSITION

SUPPOSE INITIAL STATE IS $|1\rangle = |F=1, M_F=1\rangle$

THEN FINAL STATE IS $|2\rangle = |F'=2, M_F=2\rangle$



STIMULATED EMISSION



NEW PHOTON IS IDENTICAL

ATOM LOSES ANGULAR MOMENTUM $\vec{j}$

AND ENERGY $\hbar\omega$

⇒ GOES BACK TO ORIGINAL GROUND STATE

SO TWO-LEVEL ⇒ TWO-STATE SYSTEM

$$|1\rangle = |F\,M_F\rangle$$

$$|2\rangle = |F'\,M_F'\rangle$$

$\Delta F = 0, \pm 1$ (NO $F=0 \rightarrow F'=0$)

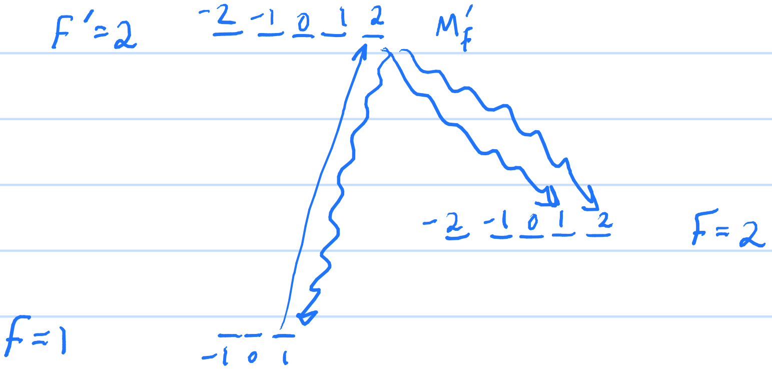
$\Delta M_F = 0, \pm 1$ DEPENDING ON POLARIZATION

SPONTANEOUS EMISSION

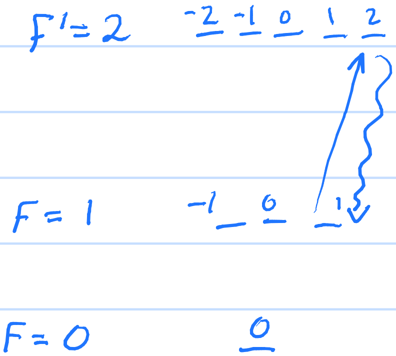
- CAN DECAY TO ANY LEVEL

BREAKS THE 2-LEVEL MODEL

e.g.



SOMETIMES STILL 2-STATE



- ONLY ONE ALLOWED TRANSITION

"CYCLING TRANSITION"

2-STATE MODEL ALSO GOOD WHEN SPONT. DECAY NEGLIGIBLE

- SHORT TIMES $t << 1/\gamma$

  i.e. MICROWAVE TRANSITION WITHIN GND STATE

  $\gamma_{M_1} \propto \omega^3 \rightarrow 0$ AS $\omega \rightarrow 0$

- NEGLIGIBLE EXCITED STATE PROB.

  i.e. LARGE DETUNING

INTERACTION W/ EXTERNAL FIELDS

- HAMILTONIAN OF ATOM IN FREE SPACE: $\hat{H}_0$

WITH APPLIED FIELDS, HAMILTONIAN IS

$$\hat{H}(t) = \hat{H}_0 + \underbrace{\hat{H}'(t)}_{\text{PERTURBATION}}$$

ELECTRIC DIPOLE INTERACTION

- FOR ATOM IN EM FIELD, FIRST-ORDER APPROX. IS:

$$H'(t) = -\vec{d} \cdot \vec{E}(t)$$

WHERE $\vec{d}$ = DIPOLE OPERATOR. FOR N ELECTRONS:

$$\vec{d} = \sum_{i=1}^{N} (-e\vec{r}_i) \qquad \text{- ASSUMING NUCLEUS AT } \vec{r} = 0$$

MAGNETIC DIPOLE INTERACTION

- INTERACTION W/ MAGNETIC FIELD $\vec{B}$

$$H'(t) = -\vec{\mu} \cdot \vec{B}(t)$$

WHERE $\vec{\mu}$ = MAGNETIC DIPOLE OPERATOR

$$= -\frac{\mu_B}{\hbar}\left(g_L\vec{L} + g_s\vec{S} + g_I\vec{I} \right)$$

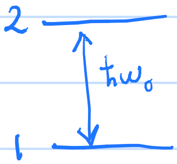
$$\approx \frac{-\mu_B}{\hbar}\left(\vec{L} + 2\vec{S} \right)$$

PHY 446 SPRING 2020
LECTURE 9

2/17/2020

- TWO-LEVEL ATOM
- FIRST-ORDER TIME-DEPENDENT SOLUTION

TWO-LEVEL ATOM



STATES $|1\rangle$ & $|2\rangle$

ENERGIES $H_0|1\rangle = E_1|1\rangle$

$H_0|2\rangle = E_2|2\rangle$

DEFINE: $\omega_1 = E_1/\hbar$, $\omega_2 = E_2/\hbar$

$\omega_0 = \omega_2 - \omega_1$ - RESONANCE FREQ.

WAVEFUNCTION: $|\psi\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle$

- c_1, c_2 CONST

SATISFIES T.D.S.E. $i\hbar\,\partial_t|\psi\rangle = H_0|\psi\rangle$

$$i\hbar\,\partial_t|\psi\rangle = i\hbar\left(c_1(-i\omega_1)e^{-i\omega_1 t}|1\rangle + c_2(-i\omega_2)e^{-i\omega_2 t}|2\rangle\right)$$

$$= \hbar\omega_1 c_1 e^{-i\omega_1 t}|1\rangle + \hbar\omega_2 c_2 e^{-i\omega_2 t}|2\rangle$$

$$= H_0|\psi\rangle$$

EXTERNAL FIELD

$$H(t) = H_0 + H'(t)$$

ELECTRIC DIPOLE INTERACTION

$$H' = -\vec{d}\cdot\vec{E}(t)$$

WHERE $\vec{E}(t) = \vec{E}(\vec{r}=0, t)$

VALID WHEN $kr \ll 1$

$$\underbrace{\left(\frac{2\pi}{\lambda}\right)(a_0)}_{\sim 10^{-4} \text{ FOR VISIBLE LIGHT}} \ll 1$$

EXPECTATION VALUE OF H' IS ZERO:

$$\langle 1|H'|1\rangle = -\underbrace{\langle 1|\vec{d}|1\rangle}_{\text{ZERO BY SYMMETRY}}\cdot\vec{E}(t) = 0$$

SAME FOR $|2\rangle$,

$$\langle 2|H'|2\rangle = 0$$

TIME EVOLUTION

$$i\hbar\,\partial_t|\Psi\rangle = H(t)\,|\Psi\rangle \qquad \text{(TDSE)}$$

LET $$|\Psi(t)\rangle = \underbrace{c_1(t)}_{\text{CHANGES DUE TO } H'}\;\underbrace{e^{-i\omega_1 t}}_{\text{THE BORING PART}}|1\rangle + c_2(t)\,e^{-i\omega_2 t}|2\rangle$$

PLUG INTO T.D.S.E.

$$\text{LHS} = i\hbar\,\partial_t|\Psi\rangle = i\hbar\Big[(\dot{c}_1 - i\omega_1 c_1)\,e^{-i\omega_1 t}|1\rangle$$
$$(\dot{c}_2 - i\omega_2 c_2)\,e^{-i\omega_2 t}|2\rangle\Big]$$
$$= (i\hbar\dot{c}_1 + \cancel{\hbar\omega_1 c_1})\,e^{-i\omega_1 t}|1\rangle$$
$$+ (i\hbar\dot{c}_2 + \cancel{\hbar\omega_2 c_2})\,e^{-i\omega_2 t}|2\rangle$$

$$\text{RHS} = (H_0 + H')|\Psi\rangle = H_0|\Psi\rangle + H'|\Psi\rangle$$
$$= \cancel{c_1 e^{-i\omega_1 t}\,\hbar\omega_1|1\rangle} + \cancel{c_2 e^{-i\omega_2 t}\,\hbar\omega_2|2\rangle}$$
$$+ c_1 e^{-i\omega_1 t} H'|1\rangle + c_2 e^{-i\omega_2 t} H'|2\rangle$$

MULTIPLY BY $\langle 1|$

$$i\hbar\,\dot{c}_1\,e^{-i\omega_1 t} = c_1 e^{-i\omega_1 t}\cancelto{0}{\langle 1|H'|1\rangle} + c_2 e^{-i\omega_2 t}\langle 1|H'|2\rangle$$

$$i\hbar\,\dot{c}_1 = e^{-i(\omega_2-\omega_1)t}\langle 1|H'(t)|2\rangle\,c_2$$
$$= e^{-i\omega_0 t}\langle 1|H'(t)|2\rangle\,c_2$$

LIKEWISE, MULTIPLY BY $\langle 2|$:

$$i\hbar\,\dot{c}_2 = e^{i\omega_0 t}\langle 2|H'(t)|1\rangle\,c_1$$

LINEARLY POLARIZED LIGHT

$$\vec{E} = E_0 \hat{x} \cos(\omega t)$$

$$H' = -\vec{d} \cdot \vec{E} = -d_x E_0 \cos\omega t$$

$$\langle 1 | H'(t) | 2 \rangle = \langle 1 | -d_x E_0 \cos(\omega t) | 2 \rangle$$

$$= -E_0 \langle 1 | d_x | 2 \rangle \cos(\omega t)$$

$$\equiv \hbar \Omega \cos(\omega t)$$

RABI FREQUENCY

$$\Omega = -\frac{1}{\hbar} E_0 \langle 1 | d_x | 2 \rangle$$

TDSE:

$$\begin{cases} i \dot{c}_1 = \Omega \cos(\omega t) e^{-i\omega_0 t} c_2 \\ i \dot{c}_2 = \Omega^* \cos(\omega t) e^{i\omega_0 t} c_1 \end{cases}$$

FIRST-ORDER SOLUTION

- CONSIDER $C_1(0) = 1$, $C_2(0) = 0$

  – SUDDEN TURN-ON OF FIELD

- FOR SHORT TIMES, WEAK FIELD, OR LARGE DETUNING

  → SMALL EXCITATION PROBABILTY $|C_2|^2$

APPROXIMATE: $C_1(t) \approx 1$; $C_2(t) \approx 0$

$$i\dot{c}_2 = \Omega^* \cos(\omega t)\, e^{i\omega_0 t}\, \overset{\approx 1}{c_1}$$

$$\approx \frac{\Omega^*}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) e^{i\omega_0 t}$$

$$= \frac{\Omega^*}{2}\left(\underbrace{e^{i(\omega+\omega_0)t}}_{\text{"COUNTER-ROTATING" TERM}} + \underbrace{e^{i(\omega_0-\omega)t}}_{\text{"CO-ROTATING" TERM}}\right)$$

INTEGRATE:

$$c_2(t) = \int_0^t \dot{c}_2\, dt + \overbrace{c_2(0)}^{0}$$

$$= -\frac{i}{2}\Omega^* \int_0^t \left(e^{i(\omega+\omega_0)t'} + e^{i(\omega_0-\omega)t'}\right) dt'$$

$$= \frac{-i\Omega^*}{2}\left[\frac{e^{i(\omega+\omega_0)t} - 1}{i(\omega+\omega_0)} + \frac{e^{i(\omega_0-\omega)t} - 1}{i(\omega_0-\omega)}\right]$$

(HW 2, PROBLEM 4 USES THIS)

- EQUIVALENT TO LORENTZ OSCILLATOR w/ $\gamma = 0$

NEAR-RESONANCE APPROX.

$$|\omega - \omega_0| \ll \omega_0$$

⇒ CO-ROTATING TERM DOMINATES

"ROTATING WAVE APPROXIMATION"

NEGLECT COUNTER-ROTATING TERM

$$C_2(t) \approx -\frac{\Omega^*}{2} \frac{e^{i(\omega_0-\omega)t}-1}{(\omega_0-\omega)}$$

$$\Delta = \omega - \omega_0$$

EXCITATION PROB.

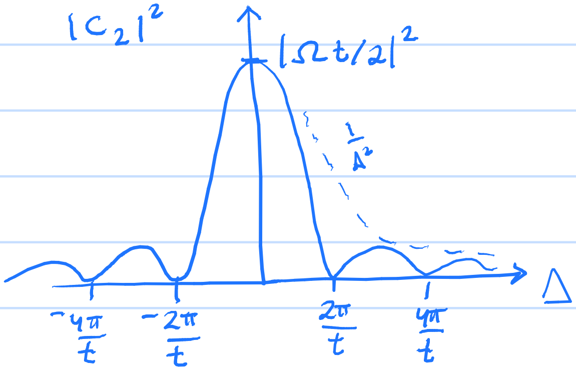
$$|C_2(t)|^2 \approx \left|\frac{\Omega}{\Delta}\right|^2 \left|\frac{e^{-i\Delta t}-1}{2}\right|^2$$

$$= \left|\frac{\Omega}{\Delta}\right|^2 \left|e^{-i\Delta t/2}\frac{e^{-i\Delta t/2}-e^{i\Delta t/2}}{2}\right|^2$$

$$= \left|\frac{\Omega}{\Delta}\right|^2 \sin^2\left(\frac{\Delta}{2}t\right)$$

VALID WHEN $|C_2| << 1$

FOR GIVEN t



MAIN POINTS:

1) $|C_2|^2 \propto |\Omega|^2 \propto |\langle 1|d_x|2\rangle|^2$

$\Rightarrow$ ALLOWED TRANSITIONS HAVE $\langle 1|d_x|2\rangle \neq 0$

ENCODES SELECTION RULES

2) $|C_2|^2$ LARGEST NEAR RESONANCE

DIPOLE MOMENT

$$|\psi\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle$$

$$\langle d_x\rangle = \langle \psi | d_x | \psi\rangle = \left(c_1^* e^{i\omega_1 t}\langle 1| + c_2^* e^{i\omega_2 t}\langle 2|\right) d_x \left(c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle\right)$$

$$= c_1^* c_2 \, e^{i(\omega_1-\omega_2)t}\langle 1|d_x|2\rangle + c.c.$$

$$= 2\,\mathrm{Re}\left[c_1^* c_2 \, e^{-i\omega_0 t}\langle 1|d_x|2\rangle\right]$$

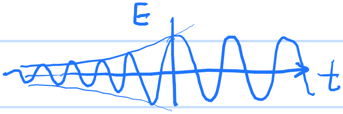
— END OF LECTURE —

SKIP

POLARIZABILITY

CONSIDER ADIABATIC RAMP OF FIELD, $|\Omega| \ll |\Delta|$
(TO REACH STEADY STATE w/o DAMPING)

$$\vec{E}(t) = \begin{cases} e^{\Gamma t} E_0 \hat{x} \cos(\omega t), & t < 0 \\ E_0 \hat{x} \cos(\omega t), & t > 0 \end{cases}$$



• SLOW RAMP: $\Gamma \ll |\Delta|$

INITIAL COND. $C_1(-\infty) = 1$, $C_2(-\infty) = 0$

• ROTATING WAVE APPROX ($|\Delta| \ll \omega_0$) FOR SIMPLICITY

FOR $t > 0$:

$$C_2(t) \approx -\frac{i}{2}\Omega^* \left[\int_{-\infty}^{0} \left(e^{\Gamma t} e^{i(\omega_0-\omega)t'} \right) dt' + \int_0^t e^{i(\omega_0-\omega)t'} dt' \right]$$

$$= -\frac{i}{2}\Omega^* \left[\frac{1-0}{\cancel{\Gamma}-i\Delta} + \frac{e^{-i\Delta t}-1}{-i\Delta} \right]$$

$$\approx -\frac{i}{2}\Omega^* \left(\frac{e^{-i\Delta t}}{-i\Delta} \right) = \frac{\Omega^*}{2\Delta} e^{-i\Delta t}$$

PLUG IN FOR $\langle d_x \rangle$:

SKIP

$$\langle d_x \rangle = 2\,\mathrm{Re}\left[c_1^* c_2\, e^{-i\omega_0 t} \langle 1|d_x|2\rangle\right]$$

$$\simeq 2\,\mathrm{Re}\left[\frac{\Omega^*}{2\Delta}\, e^{-i\Delta t}\, e^{-i\omega_0 t} \langle 1|d_x|2\rangle\right]$$

$$\left(\Omega^* = -\frac{1}{\hbar} E_0 \langle 2|d_x|1\rangle\right)$$

$$= \mathrm{Re}\left[-\frac{E_0}{\hbar} \frac{\langle 2|d_x|1\rangle}{\Delta}\, e^{-i(\omega-\omega_0)t}\, e^{-i\omega_0 t} \langle 1|d_x|2\rangle\right]$$

$$= -\frac{E_0}{\hbar\Delta}\,\mathrm{Re}\left[|\langle 1|d_x|2\rangle|^2\, e^{-i\omega t}\right]$$

$$= \boxed{-\frac{E_0}{\hbar\Delta} |\langle 1|d_x|2\rangle|^2 \cos(\omega t)} \equiv \alpha E_0 \cos(\omega t)$$

POLARIZABILITY (α)

$$\alpha(\Delta) \simeq -\frac{|\langle 1|d_x|2\rangle|^2}{\hbar\Delta}$$

LORENTZ OSCILLATOR ($\gamma = 0$):

$$\alpha_{cl}(\Delta) \simeq \frac{-e^2}{2m\omega_0} \frac{1}{\Delta}$$

(NEAR-RESONANCE APPROX.)

OSCILLATOR STRENGTH

$$\alpha(\Delta) \simeq -\frac{f_{12}\, e^2}{2m\omega_0} \frac{1}{\Delta}$$

$$\frac{f_{12}\, e^2}{2m\omega_0} = \frac{|\langle 1|d_x|2\rangle|^2}{\hbar} \qquad \longrightarrow$$

$$f_{12} = \frac{2m\omega_0}{e^2\hbar} \left|\langle 1|d_x|2\rangle\right|^2 \leq 1$$

"STRONG" TRANSITIONS : $f_{12} \sim 1$

SKIP:

CIRCULARLY POLARIZED

$$\vec{E} = \text{Re}\left[E_0\, \hat{\varepsilon}_1\, e^{-i\omega t}\right], \quad \hat{\varepsilon}_1 = \frac{-1}{\sqrt{2}}(\hat{x}+i\hat{y})$$

$$= \frac{-E_0}{\sqrt{2}} \text{Re}\Big[\underbrace{(\hat{x}+i\hat{y})(\cos\omega t - i\sin\omega t)}_{\hat{x}\cos\omega t + \hat{y}\sin\omega t}\Big]$$

$$= -\frac{E_0}{\sqrt{2}}\left(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right)$$

$$H' = -\vec{d}\cdot\vec{E} = \vec{d}\cdot\frac{E_0}{\sqrt{2}}\left(\hat{x}\cos(\omega t)+\hat{y}\sin(\omega t)\right)$$

$$= \frac{E_0}{\sqrt{2}}\left(d_x\cos(\omega t) + d_y\sin(\omega t)\right)$$

$$= \frac{E_0}{\sqrt{2}}\left[d_x\,\frac{1}{2}(e^{i\omega t}+e^{-i\omega t}) + d_y\,\frac{-i}{2}(e^{i\omega t}-e^{-i\omega t})\right]$$

$$= \frac{E_0}{2\sqrt{2}}\left[e^{i\omega t}(d_x-id_y) + e^{-i\omega t}(d_x+id_y)\right]$$

$$= \frac{E_0}{2}\left[e^{i\omega t}d_- - e^{-i\omega t}d_+\right]$$

$$d_- = \frac{1}{\sqrt{2}}(d_x - id_y)$$

$$d_+ = \frac{-1}{\sqrt{2}}(d_x + id_y)$$

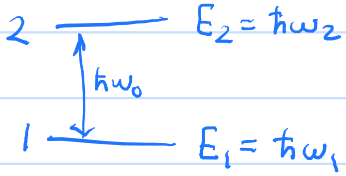
PHY 446 SPRING 2020

LECTURE 10

2/19/2020

- QUIZ
- RABI OSCILLATION

LAST TIME: 2-LEVEL ATOM



$\omega_0 = \omega_2 - \omega_1$

APPLIED FIELD : $\vec{E}(t) = E_0 \hat{x} \cos(\omega t)$

$$|\psi(t)\rangle = c_1(t) e^{-i\omega_1 t} |1\rangle + c_2(t) e^{-i\omega_2 t} |2\rangle$$

RABI FREQ : $\Omega \equiv \frac{-E_0}{\hbar} \langle 1 | d_x | 2 \rangle$

$\uparrow$ DIPOLE OPERATOR

SCHRÖDINGER EQN:

$$\begin{cases} i\dot{c}_1 = \Omega \cos(\omega t) e^{-i\omega_0 t} c_2 \\ i\dot{c}_2 = \Omega^* \cos(\omega t) e^{i\omega_0 t} c_1 \end{cases}$$

LAST TIME: SOLVED APPROXIMATELY

FOR LOW EXCITATION PROBABILITY $c_1 \approx 1$, $c_2 \approx 0$

$$\Rightarrow |c_2(t)|^2 \approx \left|\frac{\Omega}{\delta}\right|^2 \sin^2\left(\frac{\delta}{2} t\right)$$

- WORKS FOR LARGE DETUNING $|\delta| \gg |\Omega|$
- 1ST-ORDER IN TIME DEP. PERT. THEORY

RABI OSCILLATION - DIRECT SOLUTION

NOW: SOLVE TO HIGHER ORDER

APPLY ROTATING WAVE APPROXIMATION

$$i\dot{c}_1 = \frac{\Omega}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)e^{-i\omega_0 t}\; c_2$$

$$= \frac{\Omega}{2}\left(e^{i(\omega-\omega_0)t} + \underbrace{e^{-i(\omega+\omega_0)t}}_{\text{NEGLECT (OSCILLATES TOO FAST)}}\right) c_2$$

$$\approx \frac{\Omega}{2} e^{i(\omega-\omega_0)t}\; c_2 = \frac{\Omega}{2} e^{i\delta t} c_2$$

↑ ROTATING WAVE APPROX.

WHERE $\delta = \omega - \omega_0$

SIMILAR FOR $\dot{c}_2$:

$$\begin{cases} i\dot{c}_1 = \frac{\Omega}{2} e^{i\delta t} c_2 \\ i\dot{c}_2 = \frac{\Omega^*}{2} e^{-i\delta t} c_1 \end{cases}$$

COMBINE:

$$\ddot{c}_2 = -\frac{i\Omega^*}{2}\left(-i\delta\, c_1 + \dot{c}_1\right)e^{-i\delta t}$$

$$= -\,\delta\, \underbrace{\frac{\Omega^*}{2} e^{-i\delta t} c_1}_{i\dot{c}_2} - \frac{\Omega^*}{2} e^{-i\delta t} \frac{\Omega}{2} e^{i\delta t}\; c_2$$

$$= -i\delta\dot{c}_2 - \left|\frac{\Omega}{2}\right|^2 c_2$$

$$\boxed{\ddot{c}_2 + i\delta\dot{c}_2 + \left|\frac{\Omega}{2}\right|^2 c_2 = 0}$$

SOLVE LINEAR ODE:

$$\ddot{C}_2 + i\delta \dot{C}_2 + \left|\frac{\Omega}{2}\right|^2 C_2 = 0$$

SOLUTION: $C_2 \sim e^{\lambda t}$, λ = UNKNOWN

$$\lambda^2 + i\delta\lambda + \left|\frac{\Omega}{2}\right|^2 = 0$$

$$\lambda = \frac{-i\delta \pm \sqrt{-\delta^2 - 4|\Omega/2|^2}}{2}$$

$$= \frac{-i\delta \pm i\sqrt{\delta^2 + |\Omega|^2}}{2} \equiv \frac{-i\delta}{2} \pm i\frac{W}{2}$$

$$W = \sqrt{\delta^2 + |\Omega|^2}$$

INITIAL CONDITION $C_2(0) = 0$:

$$C_2(t) = e^{-i\delta t/2} A \sin(Wt/2)$$

FIND C_1 :

$$i\dot{c}_2 = \frac{\Omega^*}{2} e^{-i\delta t} C_1$$

$$\Rightarrow C_1(t) = \frac{2i}{\Omega^*} e^{i\delta t} \dot{C}_2$$

$$\dot{C}_2 = A e^{-i\delta t/2}\left[-\frac{i\delta}{2}\sin\left(\frac{Wt}{2}\right) + \frac{W}{2}\cos\left(\frac{Wt}{2}\right)\right]$$

$$C_1 = \frac{\cancel{2}i}{\Omega^*} e^{i\delta t/2} A\left[\frac{-i\delta}{\cancel{2}}\sin\left(\frac{Wt}{2}\right) + \frac{W}{\cancel{2}}\cos\left(\frac{Wt}{2}\right)\right]$$

FIND A BY NORMALIZING:

$$|C_1|^2 = \frac{1}{|\Omega|^2}|A|^2\left[\delta^2\sin^2\left(\frac{Wt}{2}\right) + W^2\cos^2\left(\frac{Wt}{2}\right)\right]$$

$$1 = |C_1|^2 + |C_2|^2 = |A|^2\left[\left(1+\frac{\delta^2}{|\Omega|^2}\right)\sin^2\left(\frac{Wt}{2}\right) + \frac{\delta^2+|\Omega|^2}{|\Omega|^2}\cos^2\left(\frac{Wt}{2}\right)\right]$$

$$= |A|^2\frac{W^2}{|\Omega|^2} \Rightarrow |A|^2 = \frac{|\Omega|^2}{W^2}$$

LET $A = -i\Omega^*/W$

PUT IT ALL TOGETHER:

SOLUTION (FOR $C_2(0)=0$, $C_1(0)=1$)

$$\begin{cases} C_1(t) = \dfrac{1}{W}e^{i\delta t/2}\left[-i\delta\sin\left(\dfrac{Wt}{2}\right) + W\cos\left(\dfrac{Wt}{2}\right)\right] \\ C_2(t) = -i\dfrac{\Omega^*}{W}e^{-i\delta t/2}\sin\left(\dfrac{Wt}{2}\right) \end{cases}$$

$$W = \sqrt{|\Omega|^2+\delta^2}$$

EXAMPLE: LET $\delta = 0$, Ω Real

APPLY LIGHT PULSE FOR TIME T, WHERE $\Omega T = \pi$

- CALLED A "π PULSE"

IF $|\psi(0)\rangle = |1\rangle$, FIND $|\psi(T)\rangle$.

$$C_1(T) = \cos(\Omega T/2) = \cos(\pi/2) = 0$$

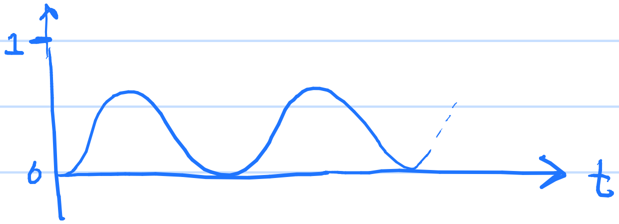
$$C_2(T) = -i\sin(\Omega T/2) = -i\sin(\pi/2) = -i$$

$$|\psi(T)\rangle = C_2(T)e^{-i\omega_2 T}|2\rangle = \boxed{-i e^{-i\omega_2 T}|2\rangle}$$

RABI OSCILLATION

EXCITATION PROBABILITY OSCILLATES

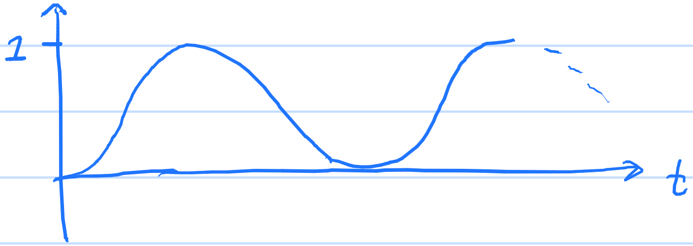
$$|c_2(t)|^2 = \frac{|\Omega|^2}{|\Omega|^2+\delta^2} \sin^2\left(\tfrac{1}{2}\sqrt{|\Omega|^2+\delta^2}\; t\right)$$



- MAX OF $|c_2|^2$ IS $\frac{|\Omega|^2}{|\Omega|^2+\delta^2}$

ON RESONANCE ($\delta = 0$):

$$|c_2(t)|^2 = \sin^2\left(\tfrac{1}{2}|\Omega| t\right)$$



- MAX $|c_2|^2$ is 1
- "RABI OSCILLATION"
- OBSERVABLE WHEN $\gamma \approx 0$ (NEGLIGIBLE DAMPING)
  - i.e. MICROWAVE TRANSITIONS & NARROW OPTICAL TRANSITIONS
- BUILDING BLOCK OF ATOMIC CLOCKS & QUANTUM COMPUTERS
- DIFFERENT FROM OSCILLATION OF DIPOLE MOMENT

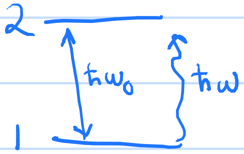
PHY 446 SPRING 2020

LECTURE 11

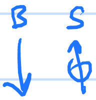
2/24/2020

- BLOCH VECTOR
- PRECESSION

TWO-LEVEL ATOM ⟷ SPIN $\frac{1}{2}$ IN B FIELD



vs.



$|1\rangle, |2\rangle$ $\qquad$ $|\uparrow\rangle, |\downarrow\rangle$

TODAY: STUDY SPIN $\frac{1}{2}$ TO GET INTUITION

WARM UP

SPIN $1/2$ $\qquad |\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$

SPIN VECTOR

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \quad ; \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

FIND $\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle$ IN TERM OF c_1, c_2

PAULI MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\Big[|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|\Big]$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

$$\langle \sigma_x \rangle = (c_1^*, c_2^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (c_1^*, c_2^*)\begin{pmatrix} c_2 \\ c_1 \end{pmatrix}$$

$$= c_1^* c_2 + c_2^* c_1 = 2\mathrm{Re}(c_1^* c_2)$$

$$\langle \sigma_y \rangle = (c_1^*, c_2^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = i(c_1^*, c_2^*)\begin{pmatrix} -c_2 \\ c_1 \end{pmatrix}$$

$$= i(c_2^* c_1 - c_1^* c_2) = 2\,\mathrm{Im}(c_1^* c_2)$$

$$\langle \sigma_z \rangle = (c_1^*, c_2^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (c_1^*, c_2^*)\begin{pmatrix} c_1 \\ -c_2 \end{pmatrix}$$

$$= |c_1|^2 - |c_2|^2$$

NOTE: $\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1 \quad (\ell)$

- PROVE FOR HW

$$\left(\mathrm{Re}(c_1^* c_2)^2 + \mathrm{Im}(c_1^* c_2)^2 = |c_1^* c_2|^2 = |c_1|^2 |c_2|^2 \right)$$

$$\ell = 4|c_1^* c_2|^2 + (|c_1|^2 - |c_2|^2)^2$$

$$= 4|c_1|^2|c_2|^2 + |c_1|^4 + |c_2|^4 - 2|c_1|^2|c_2|^2$$

$$= |c_1|^4 + |c_2|^4 + 2|c_1|^2|c_2|^2$$

$$= (|c_1|^2 + |c_2|^2)^2 = 1$$

BLOCH SPHERE

DEF: $u = \langle \sigma_x \rangle = 2\,\mathrm{Re}(c_1^* c_2)$

$v = \langle \sigma_y \rangle = 2\,\mathrm{Im}(c_1^* c_2)$

$w = \langle \sigma_z \rangle = |c_1|^2 - |c_2|^2$

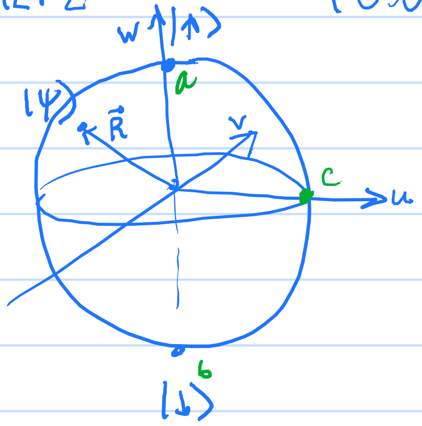
BLOCH VECTOR: $\vec{R} = (u, v, w)$

- CLASSICAL VECTOR (NOT OPERATOR)

NORM: $\vec{R}\cdot\vec{R} = u^2 + v^2 + w^2 = 1$

$(= \langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2)$

BLOCH SPHERE (UNIT SPHERE)



EX. LOCATE ON BLOCH SPHERE

a) $|\psi\rangle = |\uparrow\rangle$: $c_1 = 1,\ c_2 = 0$

$u = 0 = v,\quad w = 1$ NORTH POLE

b) $|\psi\rangle = |\downarrow\rangle$: $c_1 = 0,\ c_2 = 1$

$u = 0 = v\ ;\ w = -1$ SOUTH POLE

c) $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$: $c_1 = 1/\sqrt{2} = c_2$

$u = 1,\ v = 0,\ w = 0$. U AXIS

EVERY $|\psi\rangle$ FOR SPIN $\frac{1}{2}$ MAPS TO A POINT ON BLOCH SPHERE

NOT FULLY UNIQUE: $|\psi\rangle$ AND $e^{i\alpha}|\psi\rangle$ MAP TO SAME POINT

$\langle \sigma_x \rangle' = \langle \psi | e^{-i\alpha} \sigma_x e^{i\alpha} | \psi \rangle = \langle \psi | \sigma_x | \psi \rangle = \langle \sigma_x \rangle$

ETC

THAT'S $|\psi\rangle \to \vec{R}$. NEXT: $\vec{R} \to |\psi\rangle$

$(u, v, w) \to$ SPHERICAL COORDINATES

$$u = \sin\theta \cos\phi$$
$$v = \sin\theta \sin\phi$$
$$w = \cos\theta$$

CAN CHECK: $|\psi\rangle = e^{i\alpha}\left[\cos(\frac{\theta}{2})|\uparrow\rangle + e^{i\phi} \sin(\frac{\theta}{2})|\downarrow\rangle\right]$

ARB. PHASE

(HW)

$$c_1^* c_2 = \cos\frac{\theta}{2} \sin\frac{\theta}{2} e^{i\phi} = \frac{1}{2} \sin\theta \, e^{i\phi}$$

$$u = 2\mathrm{Re}(c_1^* c_2) = \sin\theta \cos\phi$$
$$v = 2\mathrm{Im}(c_1^* c_2) = \sin\theta \sin\phi$$
$$w = |c_1|^2 - |c_2|^2 = \cos^2(\theta/2) - \sin^2(\frac{\theta}{2})$$
$$= \cos(\frac{\theta}{2} - (-\frac{\theta}{2})) = \cos\theta$$

TIME EVOLUTION

SPIN $\frac{1}{2}$ IN B FIELD

$$H = -\vec{\mu}\cdot\vec{B} = \frac{2\mu_B}{\hbar}\vec{S}\cdot\vec{B} = \mu_B\,\vec{\sigma}\cdot\vec{B}$$

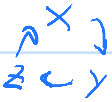
TDSE: $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$

$$\Rightarrow\ i\hbar\frac{d}{dt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mu_B \underbrace{\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & B_z \end{pmatrix}}_{\rightarrow \text{ SHOW USING } \sigma_i}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

FIND $\frac{d}{dt}\vec{R}$, USE $\frac{d}{dt}\langle\sigma_x\rangle = \left\langle\frac{i}{\hbar}\left[H,\sigma_x\right]\right\rangle$

$$[S_x, S_y] = i\hbar S_z \Rightarrow \left[\frac{\hbar}{2}\sigma_x, \cancel{\frac{\hbar}{2}}\sigma_y\right] = i\cancel{\hbar}\,\cancel{\frac{\hbar}{2}}\sigma_z$$

$$\Rightarrow [\sigma_x, \sigma_y] = 2i\sigma_z$$



$$[H, \sigma_x] = \mu_B\left[\overset{0}{\cancel{\sigma_x B_x}} + \sigma_y B_y + \sigma_z B_z, \sigma_x\right]$$

$$= 2i\mu_B\left(-\sigma_z B_y + \sigma_y B_z\right) = 2i\,\mu_B\left(\vec{\sigma}\times\vec{B}\right)_x$$

SIMILAR FOR σ_y, σ_z : $[H, \sigma_k] = 2i\,\mu_B\left(\vec{\sigma}\times\vec{B}\right)_k$

$$\Rightarrow [H, \vec{\sigma}] = 2i\,\mu_B(\vec{\sigma}\times\vec{B})$$

$$\frac{d}{dt}\langle\vec{\sigma}\rangle = \left\langle -\frac{2\mu_B}{\hbar}\,\vec{\sigma}\times\vec{B}\right\rangle$$

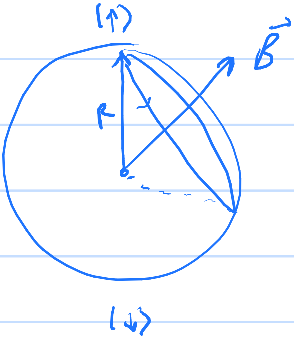
$$= \frac{2\mu_B}{\hbar}\vec{B}\times\langle\vec{\sigma}\rangle$$

$$\boxed{\frac{d}{dt}\vec{R} = \frac{2\mu_B}{\hbar}\vec{B}\times\vec{R}} \leftarrow$$

ORTHOGONAL TO $\vec{R}$ AND $\vec{B}$

· $\vec{R}\cdot\vec{R}$ IS CONSTANT

BLOCH VECTOR PRECESSES ABOUT $\vec{B}$



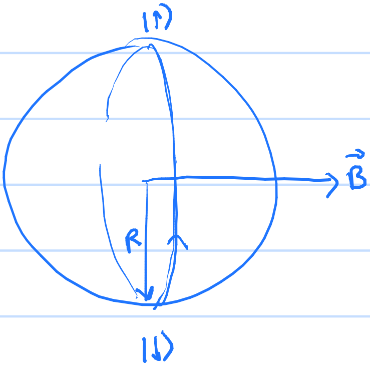
$$\frac{d}{dt}\vec{R} = \frac{2}{\hbar}\mu_B \vec{B} \times \vec{R}$$

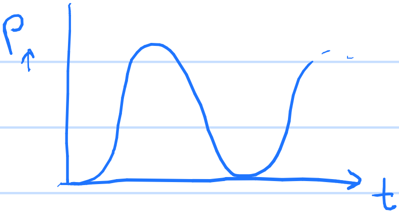
RABI OSCILLATION

SUPPOSE $\vec{B} = (B_x, 0, 0)$

AND $|\psi(0)\rangle = |\downarrow\rangle$

$\Rightarrow$ $\vec{R}(0) = (0, 0, -1)$





TWO-LEVEL ATOM

SCHRÖDINGER EQN:

$$\begin{cases} i\dot{c}_1 = \Omega \cos(\omega t) e^{-i\omega_0 t} c_2 \\ i\dot{c}_2 = \Omega^* \cos(\omega t) e^{i\omega_0 t} c_1 \end{cases}$$

ROTATING WAVE APPROX. (RWA):

$$\begin{cases} i\dot{c}_1 \approx \frac{\Omega}{2} e^{i\delta t} c_2 \\ i\dot{c}_2 \approx \frac{\Omega^*}{2} e^{-i\delta t} c_1 \end{cases}$$

$$\delta \equiv \omega - \omega_0$$

NEXT: SIMPLIFY USING CHANGE OF VARIABLES

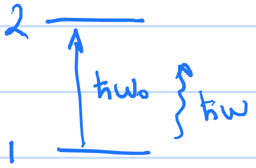
MAP ONTO BLOCH SPHERE MODEL

PHY 446 SPRING 2020
LECTURE 12

2/26/2020

- ROTATING FRAME TRANSFORMATION
- BLOCH VECTOR PRECESSION

TWO-LEVEL ATOM



$$|\psi\rangle = c_1 e^{-i\omega_1 t}|1\rangle + c_2 e^{-i\omega_2 t}|2\rangle$$

$$\omega_0 = \omega_2 - \omega_1$$

– NEGLECTING SPONT. EMISS.

ROTATING WAVE APPROX. (RWA):

$$\begin{cases} i\dot{c}_1 \approx \frac{\Omega}{2} e^{i\delta t} c_2 \\ i\dot{c}_2 \approx \frac{\Omega^*}{2} e^{-i\delta t} c_1 \end{cases}$$

$$\delta \equiv \omega - \omega_0$$

ELIMINATE TIME-DEPENDENCE USING "ROTATING FRAME TRANSFORMATION"

$$\tilde{c}_1 = c_1 e^{-i\delta t/2}$$

$$\tilde{c}_2 = c_2 e^{i\delta t/2}$$

$$\dot{\tilde{c}}_1 = \dot{c}_1 e^{-i\delta t/2} - \frac{i\delta}{2} c_1 e^{-i\delta t/2}$$

$$= -\frac{i\Omega}{2} e^{i\delta t/2} c_2 - i\frac{\delta}{2}\tilde{c}_1 = -\frac{i\Omega}{2}\tilde{c}_2 - i\frac{\delta}{2}\tilde{c}_1$$

$$i\dot{\tilde{c}}_2 = i\dot{c}_2 e^{i\delta t/2} - c_2 e^{i\delta t/2}\frac{\delta}{2} = \frac{\Omega^*}{2}\tilde{c}_1 - \frac{\delta}{2}\tilde{c}_2$$

PLUG IN : (RWA+RFT)

$$\begin{cases} i\dot{\tilde{c}}_1 = \frac{1}{2}(\delta \tilde{c}_1 + \Omega \tilde{c}_2) \\ i\dot{\tilde{c}}_2 = \frac{1}{2}(\Omega^* \tilde{c}_1 - \delta \tilde{c}_2) \end{cases}$$

MATRIX FORM

$$i\hbar \frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \underbrace{\frac{\hbar}{2}\begin{pmatrix}\delta & \Omega\\ \Omega^* & -\delta\end{pmatrix}}_{H_{eff}}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$$

$$H_{Eff} = \frac{\hbar}{2}\begin{pmatrix}\delta & \Omega\\ \Omega^* & -\delta\end{pmatrix} = \frac{\hbar}{2}\left[\Omega_r \sigma_x - \Omega_i \sigma_y + \delta \sigma_z\right]$$

$$= \frac{\hbar}{2}(\Omega_r, -\Omega_i, \delta)\cdot\vec{\sigma} \equiv \frac{\hbar}{2}\vec{W}\cdot\vec{\sigma}$$

RECALL: ELECTRON SPIN IN B FIELD

$$H = -\vec{\mu}\cdot\vec{B} = \mu_B \vec{B}\cdot\vec{\sigma}$$

$$\Rightarrow \quad \frac{d}{dt}\langle\vec{\sigma}\rangle = \frac{2}{\hbar}\mu_B \vec{B}\times\langle\vec{\sigma}\rangle$$

COMPONENTS $\vec{R} = \langle\vec{\sigma}\rangle = (u, v, w)$; $|\psi\rangle = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$$\begin{cases} u = 2\,\mathrm{Re}(c_1^* c_2) \\ v = 2\,\mathrm{Im}(c_1^* c_2) \\ w = |c_1|^2 - |c_2|^2 \end{cases}$$

PROTON IN B-FIELD

$$H = -\vec{\mu}\cdot\vec{B} = -\mu_p \vec{B}\cdot\vec{\sigma}$$

$$\Rightarrow \frac{d}{dt}\langle\vec{\sigma}\rangle = \frac{2}{\hbar}\mu_p\langle\vec{\sigma}\rangle\times\vec{B}$$ (LH RULE!)

TWO-LEVEL ATOM: LIKE ELECTRON

$$\mu_B\vec{B} \leftrightarrow \frac{\hbar}{2}\vec{W}$$

HISTORICAL SIGN CONVENTION: IMITATE PROTON

DEFINE BLOCH VECTOR FOR 2-LEVEL ATOM

$$\begin{cases} u = 2\,\mathrm{Re}(\tilde{c}_1^*\tilde{c}_2) \\ v = \enclose{circle}{-}\,2\,\mathrm{Im}(\tilde{c}_1^*\tilde{c}_2) \\ w = |\tilde{c}_1|^2 - |\tilde{c}_2|^2 = |c_1|^2 - |c_2|^2 \end{cases}$$

← ! GIVES LH. RULE

$$\vec{R} = (u, v, w)$$

$$\frac{d\vec{R}}{dt} = \vec{R}\times\vec{W}$$ (LH. RULE)

OR: RH Rule about $-\vec{W}$

- TYPICALLY Ω IS REAL: $\Omega_r = \Omega$; $\Omega_i = 0$

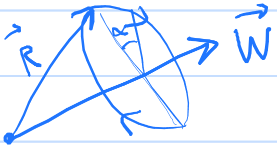
$$\vec{W} = (\Omega, 0, \delta)$$

- SOLVE FOR $|c_1|^2, |c_2|^2$: USE $|c_1|^2 + |c_2|^2 = 1$

$$w = 2|c_1|^2 - 1 = 1 - 2|c_2|^2$$

$$|c_1|^2 = \frac{1+w}{2} \quad ; \quad |c_2|^2 = \frac{1-w}{2}$$

PRECESSION : $\dot{\vec{R}} = \vec{R}\times\vec{W}$



ANGULAR VELOCITY: $\alpha = \|\vec{W}\| t = \sqrt{\Omega^2+\delta^2}\; t$

EX. $|\psi(0)\rangle = |1\rangle$; $\delta = 0$

a) DRAW $\vec{R}(0)$, $\vec{W}$ ON BLOCH SPHERE

b) DRAW $\vec{R}(t)$ PATH ON BLOCH SPHERE

c) FIND $w(t)$ AND $|c_2(t)|^2$

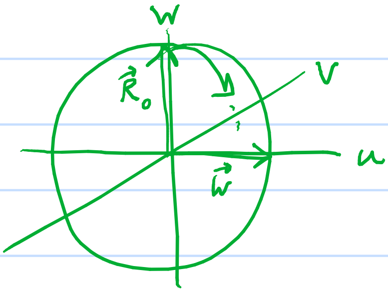
↑ LOWER CASE

- HINT: DRAW R(t) IN v,w PLANE

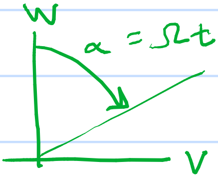
a, b)

$t=0$: $\vec{R}(0) = (0,0,1)$

$\vec{W} = (\Omega,0,0)$



c)



$w(t) = \cos\alpha = \boxed{\cos(\Omega t)}$

(PROB. OSCILLATES AT $\underline{\Omega}$)

[$v(t) = \sin(\Omega t)$]

$$|c_2(t)|^2 = \frac{1-w}{2} = \frac{1-\cos(\Omega t)}{2} = \boxed{\sin^2\left(\frac{\Omega t}{2}\right)}$$

[$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

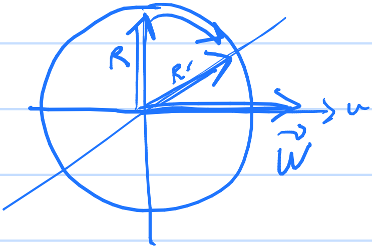
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$]

π/2 PULSE

FOR $\delta = 0$, $\alpha(t) = \Omega t$

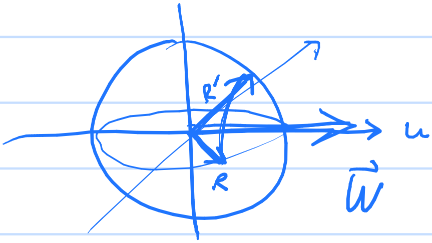
LET $\Omega T = \pi/2$

BLOCH VECTOR ROTATES BY $\alpha(T) = \pi/2$ (90°)



TRUE FOR ANY INITIAL CONDITION

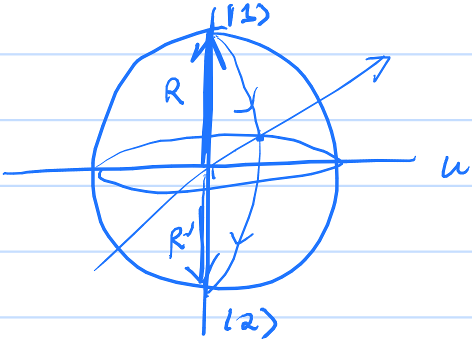
E.x



π PULSE

$\Omega T = \pi \Rightarrow$ ROTATION BY π (180°)

e.x.



FREE PRECESSION

$\Omega = 0$

$\delta \neq 0$

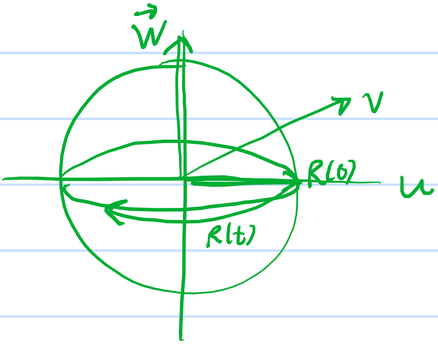
$\vec{W} = (0, 0, \delta)$

e.x. LET $\delta > 0$, $\vec{R}(t) = (1, 0, 0)$

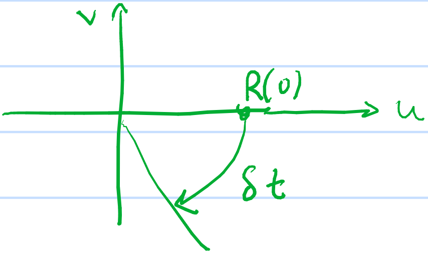
a) DRAW $\vec{W}$, $\vec{R}(0)$, $\vec{R}(t)$

b) FIND EQUATION FOR $\vec{R}(t) = (u(t), v(t), w(t))$

a)



b) u-v PLANE:



$$\alpha(t) = \|\vec{W}\| t = \delta t$$

$$\begin{cases} u(t) = \cos(\delta t) \\ v(t) = -\sin(\delta t) \\ w(t) = 0 \end{cases}$$

PHY 446 SPRING 2020

LECTURE 13

3/2/2020

- SPONTANEOUS DECAY
- OPTICAL BLOCH EQUATIONS

WARM-UP

BLOCH VECTOR $|\psi(0)\rangle = |2\rangle$

STRONG $\pi/2$ PULSE: $\Omega \gg \delta$ $(\Omega > 0, \delta > 0)$

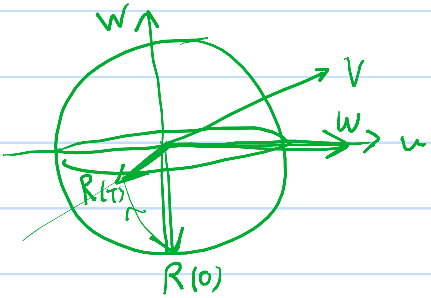
$$\Omega T = \pi/2$$

a) FIND & DRAW $\vec{R}(0)$

b) DRAW $\vec{W} = (\Omega, 0, \delta)$ FOR $\Omega \gg \delta$ LIMIT

c) FIND & DRAW $\vec{R}(T)$ USING: $\frac{d\vec{R}}{dt} = -\vec{W} \times \vec{R}$

$\vec{R}(0) = (0, 0, -1)$



$\vec{R}(T) = (-1, 0, 0)$

TWO-LEVEL ATOM

$$|\psi(t)\rangle = c_1(t)\, e^{-i\omega_1 t} |1\rangle + c_2(t)\, e^{-i\omega_2 t} |2\rangle$$

ROTATING FRAME TRANSFORMATION

$$\tilde{c}_1(t) = c_1 e^{-i\delta t/2}$$

$$\tilde{c}_2(t) = c_2 e^{i\delta t/2}$$

BLOCH VECTOR

$$\begin{cases} u = 2\,\mathrm{Re}[\tilde{c}_1^* \tilde{c}_2] = \tilde{c}_1 \tilde{c}_2^* + \tilde{c}_2 \tilde{c}_1^* \\ v = -2\,\mathrm{Im}[\tilde{c}_1^* \tilde{c}_2] = -i(\tilde{c}_1 \tilde{c}_2^* - \tilde{c}_2 \tilde{c}_1^*) \\ w = |\tilde{c}_1|^2 - |\tilde{c}_2|^2 = \tilde{c}_1 \tilde{c}_1^* - \tilde{c}_2 \tilde{c}_2^* \end{cases}$$

$$\vec{R} = (u, v, w)$$

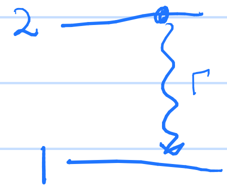
EQN. OF MOTION:

$$\begin{cases} \frac{d}{dt}\vec{R} = -\vec{W} \times \vec{R} = \vec{R} \times \vec{W} \\ \vec{W} = (\Omega, 0, \delta) \end{cases}$$

COMPONENTS:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{vmatrix} \hat{u} & \hat{v} & \hat{w} \\ u & v & w \\ \Omega & 0 & \delta \end{vmatrix} = \begin{pmatrix} \delta v \\ -\delta u + \Omega w \\ -\Omega v \end{pmatrix}$$

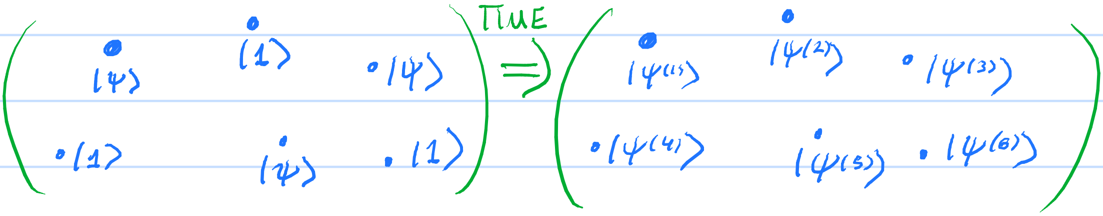
SPONTANEOUS DECAY



WVFN COLLAPSE $|\psi\rangle \to |1\rangle$

- NOT INCLUDED IN TDSE
- EMITTED PHOTON DETECTABLE

CONSIDER MANY ATOMS; SOME HAVE DECAYED



ENSEMBLE AVG BLOCH VECTOR

$$\langle \tilde{c}_i \tilde{c}_j^* \rangle_e = \sum_{k=1}^{N} \underbrace{\tilde{c}_i^{(k)} \tilde{c}_j^{(k)*}}_{\text{from k-th atom}} \frac{1}{N}$$

ensemble avg.

$$\equiv \tilde{\rho}_{ij}$$

← Rotating frame

$$\begin{cases} u = \langle \tilde{c}_1 \tilde{c}_2^* \rangle_e + \langle \tilde{c}_2 \tilde{c}_1^* \rangle_e = \tilde{\rho}_{12} + \tilde{\rho}_{21} \\ v = -i\left(\tilde{\rho}_{12} - \tilde{\rho}_{21}\right) \\ w = \tilde{\rho}_{11} - \tilde{\rho}_{22} \end{cases}$$

$$\vec{R} = (u, v, w)$$

DENSITY MATRIX $\tilde{\rho} = \begin{pmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{21} \\ \tilde{\rho}_{21} & \tilde{\rho}_{22} \end{pmatrix}$

EXCITED STATE FRACTION

$$P_2 = \langle \tilde{c}_2 \tilde{c}_2^* \rangle_e = \tilde{\rho}_{22}$$

GROUND STATE FRACTION

$$P_1 = \langle \tilde{c}_1 \tilde{c}_1^* \rangle_e = \tilde{\rho}_{11}$$

TOTAL PROB.

$$P_1 + P_2 = \tilde{\rho}_{11} + \tilde{\rho}_{22} = \sum_{k=1}^{N} \frac{1}{N} \Big(\underbrace{|\tilde{c}_1^{(k)}|^2 + |\tilde{c}_2^{(k)}|^2}_{1} \Big)$$

$$= 1$$

NOTE: $\mathrm{Tr}(\tilde{\rho}) = \tilde{\rho}_{11} + \tilde{\rho}_{22} = 1$ (NORMALIZED)

$$W = \tilde{\rho}_{11} - \tilde{\rho}_{22} = 1 - 2\tilde{\rho}_{22} = 2\tilde{\rho}_{11} - 1$$

$$\Rightarrow \tilde{\rho}_{11} = \frac{1+w}{2} \quad ; \quad \tilde{\rho}_{22} = \frac{1+w}{2}$$

PROBABILITY INTERPRETATION

ENSEMBLE → POSSIBLE STATES of ONE ATOM

(DECAY HAPPENS AT RANDOM TIMES)

$\tilde{\rho}_{ij}$ = EXPECTED VALUE of $\tilde{c}_i \tilde{c}_j^* = \langle \tilde{c}_i \tilde{c}_j^* \rangle_c$

EQN. OF MOTION FOR AVG. BLOCH VECTOR:

OPTICAL BLOCH EQNS

$$\begin{cases} \dot{u} = -\delta v - \frac{\Gamma}{2} u \\ \dot{v} = -\delta u + \Omega w - \frac{\Gamma}{2} v \\ \dot{w} = -\Omega v - \Gamma (w - 1) \end{cases}$$

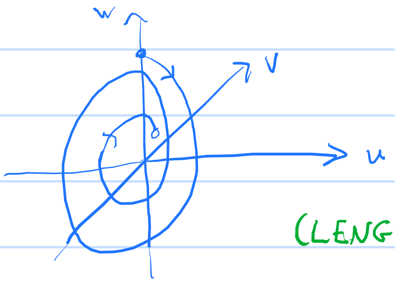
DAMPED RABI OSCILLATION

Ex. $\delta = 0$, $\Gamma \ll \Omega$, $R(0) = (0,0,1)$

$$\dot{u} = -\frac{\Gamma}{2} u \quad \Rightarrow u = 0$$

$$\begin{cases} \dot{v} = \Omega w - \frac{\Gamma}{2} v \\ \dot{w} = -\Omega v - \Gamma(w-1) \end{cases}$$

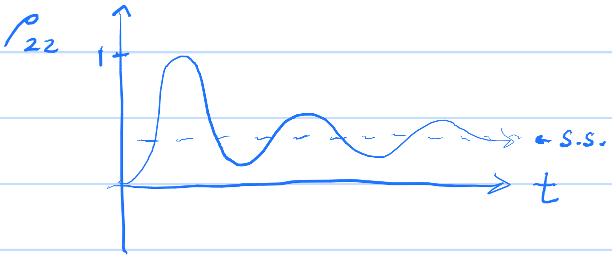
DAMPED CIRCULAR MOTION IN V-W PLANE



$\Rightarrow \vec{W}$

(LENGTH NOT CONSTANT!)

EXCITED STATE FRACTION $\tilde{\rho}_{22} = \frac{1-w}{2}$



STEADY STATE ($\dot{u} = 0, \dot{v} = 0, \dot{w} = 0$)

$$\rho_{22} \rightarrow \frac{1}{2} \frac{\Omega^2}{\Omega^2 + \Gamma^2/2} \xrightarrow{\Omega \gg \Gamma} \frac{1}{2} \qquad (\delta = 0)$$

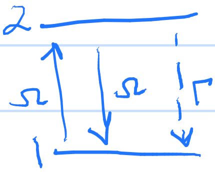
APPROX. SOLUTION IN $\Gamma \ll \Omega$ LIMIT:

$$w \approx e^{-\frac{3}{4}\Gamma t} \cos(\Omega t)$$

$$v \approx e^{-\frac{3\Gamma}{4} t} \sin(\Omega t)$$

$$\rho_{22} = \frac{1-w}{2} \approx \frac{1}{2}\left(1 - e^{-\frac{3}{4}\Gamma t} \cos(\Omega t)\right)$$

$\delta = 0$ PICTURE:



- $\rho_{22}^{(ss)} \leq \frac{1}{2}$ (MORE DOWNWARD PROCESSES THAN UP)
- $\rho_{22}^{(ss)} \to \frac{1}{2}$ AS $\frac{\Omega}{\Gamma} \to \infty$ (DECAY RATE BECOMES NEGLIGIBLE)

GENERAL STEADY STATE SOLUTION

$0 = \dot{u} = \dot{v} = \dot{w}$ GIVES:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{s.s.} = \frac{1}{\delta^2 + \frac{\Omega^2}{2} + \frac{\Gamma^2}{4}} \begin{pmatrix} \Omega\delta \\ \Omega\Gamma/2 \\ \delta^2 + \Gamma^2/4 \end{pmatrix}$$

EXCITED STATE PROB.

$$\rho_{22}^{(ss)} = \frac{1 - w_{ss}}{2} = \frac{\Omega^2/4}{\delta^2 + \Omega^2/2 + \Gamma^2/4}$$

$$= \frac{\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2 + (2\delta/\Gamma)^2}$$

$$= \frac{1}{2} \frac{S}{1 + S + (\frac{\delta}{\Gamma/2})^2}$$

$S \equiv 2\Omega^2/\Gamma^2$ "SATURATION PARAMETER"

RATE OF PHOTON SCATTERING

$$R_{sc} = \Gamma \rho_{22}^{(ss)} = \frac{\Gamma}{2} \frac{S}{1 + S + (\frac{\delta}{\Gamma/2})^2}$$

EXTRA STUFF :

$R(0) = (0,0,1)$; $\delta = 0$

$$\dot u = -\frac{\Gamma}{2} u \quad \Rightarrow u = 0$$

$$\begin{cases} \dot v = \Omega w - \frac{\Gamma}{2} v \\ \dot w = -\Omega v - \Gamma(w-1) \end{cases}$$

$$\ddot w = -\Omega \dot v - \Gamma \dot w = -\Omega\left(\Omega w - \frac{\Gamma}{2} v\right) + \Gamma\left(\Omega v + \Gamma(w-1)\right)$$

$$= -\Omega^2 w + \underbrace{\frac{1}{2}\Omega\Gamma v + \Gamma\Omega v}_{3/2\,\Omega\Gamma v} + \Gamma^2(w-1)$$

$$\Omega v = -\dot w - \Gamma(w-1)$$

$$\ddot w = -\Omega^2 w - \frac{3}{2}\Gamma\left(\dot w + \Gamma(w-1)\right) + \Gamma^2(w-1)$$

$$= -\Omega^2 w - \frac{3}{2}\Gamma\dot w - \frac{3}{2}\Gamma^2(w-1) + \frac{2}{2}\Gamma^2(w-1)$$

$$= -\Omega^2 w - \frac{3}{2}\Gamma\dot w - \frac{1}{2}\Gamma^2(w-1)$$

$$= -(\Omega^2 + \Gamma^2/2)w - \frac{3}{2}\Gamma\dot w + \Gamma^2/2$$

$$\ddot w + \frac{3}{2}\Gamma\dot w + (\Omega^2 + \Gamma^2/2)w = \Gamma^2/2$$

Particular: steady state $w_p = \text{const}$

$$(\Omega^2 + \Gamma^2/2)w_p = \Gamma^2/2 \Rightarrow w_p = \frac{\Gamma^2/2}{\Omega^2 + \Gamma^2/2}$$

Homogeneous: $w = e^{\lambda t}$

$$\lambda^2 + \frac{3}{2}\Gamma\lambda + (\Omega^2 + \Gamma^2/2) = 0$$

$$\lambda = \frac{-\frac{3}{2}\Gamma \pm \sqrt{\left(\frac{3\Gamma}{2}\right)^2 - 4(\Omega^2 + \Gamma^2/2)}}{2} \qquad \overset{1/4}{\left(\frac{9}{4} - \frac{8}{4}\right)\Gamma^2}$$

$$= -\frac{3}{4}\Gamma \pm \frac{i}{2}\sqrt{4\Omega^2 - \frac{1}{4}\Gamma^2} = -\frac{3}{4}\Gamma \pm i\sqrt{\Omega^2 - \Gamma^2/16}$$

FOR $W(0) = 1$; $\dot w(0) = -\Omega v(0) - \Gamma(w(0) - 1) = 0$

$$W(t) = w_{ss} + e^{-\frac{3}{4}\Gamma t}\left(A\cos\left[\underbrace{\sqrt{\Omega^2 - \Gamma^2/16}}_{\alpha}\, t\right] + B\sin\left[\sqrt{\Omega^2 - \Gamma^2/16}\, t\right]\right)$$

$$1 = W(0) = w_{ss} + A \Rightarrow A = 1 - w_{ss} = \frac{\Omega^2}{\Omega^2 + \Gamma^2/2}$$

$$\dot w = -\frac{3}{4}\Gamma e^{-\frac{3}{4}\Gamma t}\left(A\cos\alpha t + B\sin\alpha t\right)$$
$$+ e^{-3/4\,\Gamma t}\left(-\alpha A\sin\alpha t + \alpha B\cos\alpha t\right)$$

$$0 = \dot w(0) = -\frac{3}{4}\Gamma A + \alpha B \Rightarrow \alpha B = \frac{3}{4}\Gamma A$$

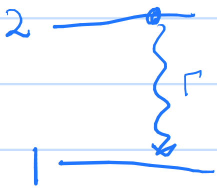
$$B = \frac{3}{4}\Gamma\frac{A}{\alpha}$$

FOR $\Omega \gg \Gamma$, $w \approx e^{-\frac{3}{4}\Gamma t}\cos(\Omega t)$

$$\rho_{22} = \frac{1-w}{2} = \frac{1}{2}\left(1 - e^{-\frac{3}{4}\Gamma t}\cos(\Omega t)\right)$$

$$v \simeq \frac{-1}{\Omega}\dot w \approx e^{-\frac{3\Gamma}{4}t}\sin(\Omega t)$$

SPONTANEOUS DECAY (HEURISTIC DERIVATION)



SUPPOSE $\Omega = 0 = \delta$

AVERAGE EFFECT: (ENSEMBLE AVG.)

$$|\tilde{c}_2|^2 \sim e^{-\Gamma t}$$

$$\tilde{c}_2 \sim e^{-\Gamma t/2}$$

$$u = 2\,\mathrm{Re}[\tilde{c}_1^* \tilde{c}_2] \sim e^{-\Gamma t/2}$$

$$\Rightarrow \dot{u} = -\tfrac{1}{2}\Gamma u$$

$$v = -2\,\mathrm{Im}[\tilde{c}_1^* \tilde{c}_2] \sim e^{-\Gamma t/2}$$

$$\Rightarrow \dot{v} = -\tfrac{1}{2}\Gamma v$$

$$|c_2|^2 = \frac{1-w}{2}$$

$$w - 1 = -2|c_2|^2 \sim e^{-\Gamma t}$$

$$\dot{w} = \frac{d}{dt}(w-1) \sim -\Gamma(w-1)$$

PURE SPONT. EMISS.:

$$\Rightarrow \begin{cases} \dot{u} = -\frac{\Gamma}{2} u \\ \dot{v} = -\frac{\Gamma}{2} v \\ \dot{w} = -\Gamma(w-1) \end{cases}$$

$$\frac{d}{dt}(u^2+v^2+w^2) = 2u\dot{u} + 2v\dot{v} + 2w\dot{w}$$

$$= -\Gamma u^2 - \Gamma v^2 - 2\Gamma(w^2 - w)$$

$$= -\Gamma \quad - \Gamma w^2 + 2\Gamma w = -\Gamma(1 + w^2 - 2w)$$

$$= -\Gamma(w-1)^2$$

$$u^2 = (\tilde{\rho}_{12} + \tilde{\rho}_{21})^2 = \tilde{\rho}_{12}^2 + 2\tilde{\rho}_{12}\tilde{\rho}_{21} + \tilde{\rho}_{21}^2$$

$$v^2 = -(\tilde{\rho}_{12} - \tilde{\rho}_{21})^2 = -\tilde{\rho}_{12}^2 + 2\tilde{\rho}_{12}\tilde{\rho}_{21} - \tilde{\rho}_{21}^2$$

$$w^2 = (\tilde{\rho}_{11} - \tilde{\rho}_{22})^2$$

DENSITY MATRIX

COMBINE CLASSICAL & QUANTUM PROBABILITY

POSSIBLE WAVEFUNCTIONS : $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_k\rangle, \ldots$

PROBABILITIES : $P_1, P_2, \ldots, P_k, \ldots$

EXPECTATION VALUES:

$$\langle A \rangle = \sum_k P_k \langle \psi_k | A | \psi_k \rangle$$

$$= \sum_{n,k} P_k \langle \psi_k | A | n \rangle \langle n | \psi_k \rangle \qquad (n = 1, 2)$$

$$= \sum_n \langle n | \underbrace{\sum_k P_k | \psi_k \rangle \langle \psi_k |}_{\rho \ (2 \times 2 \text{ MATRIX})} A | n \rangle$$

$$= \sum_n \langle n | \rho A | n \rangle = \mathrm{Tr}(\rho A)$$

$$\rho = \sum_k P_k | \psi_k \rangle \langle \psi_k |$$

$$= \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

MANY P_k

$\rightarrow$ FOUR ρ_{ij}

PURE STATE:

KNOWN WVFN $|\psi\rangle$

$$\rho = |\psi\rangle\langle\psi|$$

LET $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$

CALCULATE ρ MATRIX

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \; ; \; \langle\psi| = (c_1^*, c_2^*)$$

$$\rho = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (c_1^*, c_2^*) = \begin{pmatrix} c_1 c_1^* & c_1 c_2^* \\ c_2 c_1^* & c_2 c_2^* \end{pmatrix}$$

ROTATING FRAME (PURE STATE)

$$\tilde{\rho} = \begin{pmatrix} \tilde{c}_1 \tilde{c}_1^* & \tilde{c}_1 \tilde{c}_2^* \\ \tilde{c}_2 \tilde{c}_1^* & \tilde{c}_2 \tilde{c}_2^* \end{pmatrix} = \begin{pmatrix} |c_1|^2 & e^{-i\delta t} c_1 c_2^* \\ e^{i\delta t} c_1 c_2^* & |c_2|^2 \end{pmatrix}$$

GENERAL: "MIXED STATE" (NOT PURE)

UPGRADE $c_i c_j^* \longrightarrow \rho_{ij}$

$$\tilde{\rho} = \begin{pmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \tilde{\rho}_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} & e^{-i\delta t} \rho_{12} \\ \rho_{21} e^{i\delta t} & \rho_{22} \end{pmatrix}$$

PHY 446 SPRING 2020

LECTURE 14

3/4/2020

- STEADY-STATE SOLN. TO O.B.E.
- SATURATION

WARM-UP : 2-LEVEL ATOM AT $t=0$

50% CHANCE OF $\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

50% CHANCE OF $\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ i\,1/\sqrt{2}\end{pmatrix}$

a) FIND DENSITY MATRIX $\tilde{\rho}(t=0)$

b) FIND AVG. BLOCH VECTOR $\vec{R}(0)$

c) IF $\Omega=0$, $\delta=0$, $\Gamma\neq 0$, FIND $\vec{R}(t)$ EXPRESSION

d) SKETCH $R(t)$; SHOW $R(0)$ & $R(t\to\infty)$

a) $\tilde{\rho}_{11} = \langle \tilde{c}_1\tilde{c}_1^* \rangle = 0.5\,\frac{1}{2} + 0.5\,\frac{1}{2} = \frac{1}{2}$

$\tilde{\rho}_{22} = \langle \tilde{c}_2\tilde{c}_2^* \rangle = 0.5\,\frac{1}{2} + 0.5\,\frac{1}{2} = \frac{1}{2}$

$\tilde{\rho}_{12} = \langle \tilde{c}_1\tilde{c}_2^* \rangle = 0.5\,\frac{1}{2} - i\,0.5\,\frac{1}{2} = \frac{1}{4}(1-i)$

$\tilde{\rho}_{21} = \langle \tilde{c}_2\tilde{c}_1^* \rangle = 0.5\,\frac{1}{2} + i\,0.5\,\frac{1}{2} = \frac{1}{4}(1+i)$

b) $u = \tilde{\rho}_{12} + \tilde{\rho}_{21} = \frac{1}{2}$

$v = 2\,\mathrm{Im}(\tilde{\rho}_{12}) = -i(\tilde{\rho}_{12} - \tilde{\rho}_{21}) = -i\,(-i/2) = -\frac{1}{2}$

$w = \tilde{\rho}_{11} - \tilde{\rho}_{22} = 0$

$\vec{R}(0) = (1/2, -1/2, 0)$

c) OBE ($\Omega = 0 = \delta$)

$$\begin{cases} \dot{u} = -\frac{\Gamma}{2} u \\ \dot{v} = -\frac{\Gamma}{2} v \\ \dot{w} = -\Gamma (w-1) \end{cases}$$

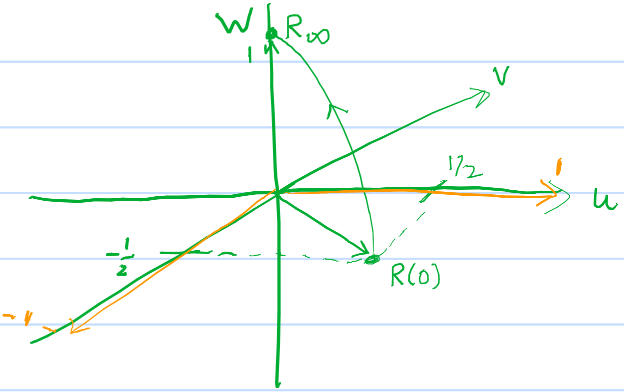
$$u(t) = u(0)\, e^{-\Gamma t/2} = \frac{1}{2} e^{-\Gamma t/2}$$

$$v(t) = v(0)\, e^{-\Gamma t/2} = -\frac{1}{2} e^{-\Gamma t/2}$$

$$w(t) = \underset{\uparrow \text{particular}}{1} + A e^{-\Gamma t} = \underbrace{1 - e^{-\Gamma t}}_{w(0) = 0 \ \checkmark}$$

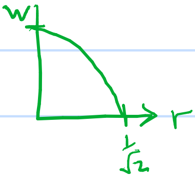
d) $R(0) = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$

$R(t \to \infty) = (0, 0, 1)$



$$u^2 + v^2 = \frac{1}{4} e^{-\Gamma t} \times 2 = \frac{1}{2} e^{-\Gamma t}$$

$$w = 1 - 2(u^2 + v^2) = 1 - 2r^2$$



NOTE: COULD FIND THE SEPARATE BLOCH VECTORS

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \rightarrow \vec{R} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \rightarrow \vec{R} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

AVG: $\langle \vec{R} \rangle = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$ ✓

OPTICAL BLOCH EQNS

$$\dot{u} = -\delta v - \frac{\Gamma}{2} u$$

$$\dot{v} = -\delta u + \Omega w - \frac{\Gamma}{2} v$$

$$\dot{w} = -\Omega v - \Gamma (w-1)$$

GENERAL STEADY STATE SOLUTION

$0 = \dot{u} = \dot{v} = \dot{w}$ GIVES:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{s.s.} = \frac{1}{\delta^2 + \frac{\Omega^2}{2} + \frac{\Gamma^2}{4}} \begin{pmatrix} \Omega \delta \\ \Omega \Gamma/2 \\ \delta^2 + \Gamma^2/4 \end{pmatrix}$$

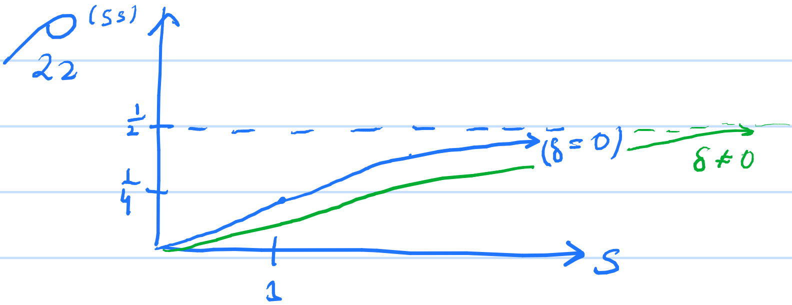
EXCITED STATE PROB.

$$\rho_{22}^{(ss)} = \frac{1 - W_{ss}}{2} = \frac{\Omega^2/4}{\delta^2 + \Omega^2/2 + \Gamma^2/4}$$

$$= \frac{\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2 + (2\delta/\Gamma)^2}$$

$$= \frac{1}{2} \frac{S}{1 + S + \left(\frac{\delta}{\Gamma/2}\right)^2}$$

$S \equiv 2\Omega^2/\Gamma^2$ "SATURATION PARAMETER"



SATURATION INTENSITY

$\Omega^2 \propto I$, so $s = I/I_{SAT}$

FOR SOME "SATURATION INTENSITY" I_{SAT}

WHAT IS I_{SAT}?

$$I_{SAT} = I/s = \frac{\Gamma^2 I}{2\Omega^2}$$

RELATE Ω^2 TO I:

$$\vec{E}(t) = E_0 \hat{x} \cos(\omega t)$$

$$\Omega^2 = \frac{e^2 |X_{12}|^2 E_0^2}{\hbar^2}$$

WHERE $X_{12} = \langle 1 | x | 2 \rangle$

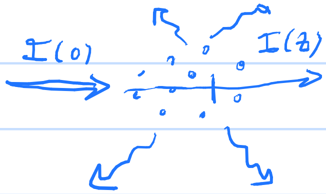
$$I = \langle \varepsilon_0 c \vec{E}^2 \rangle = \frac{1}{2} \varepsilon_0 c E_0^2$$

$$E_0^2 = \frac{2I}{\varepsilon_0 c}$$

$$\Omega^2 = \frac{e^2 |X_{12}|^2}{\hbar^2} \frac{2I}{\varepsilon_0 c} \rightarrow \frac{I}{\Omega^2} = \frac{\hbar^2 \varepsilon_0 c}{2 e^2 |X_{12}|^2}$$

$$I_{SAT} = \frac{\Gamma^2 \hbar^2 \varepsilon_0 c}{4 e^2 |X_{12}|^2}$$

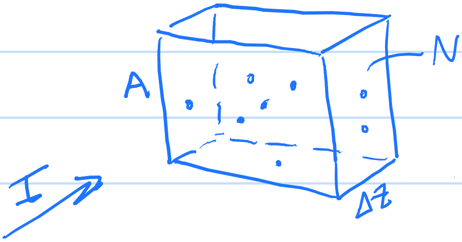
ABSORPTION



RATE OF PHOTON SCATTERING (per atom)

$$\frac{R_{sc}}{N} = \Gamma \rho_{22}^{(ss)} = \frac{\Gamma}{2} \frac{I/I_{SAT}}{1+S+\left(\frac{\delta}{\Gamma/2}\right)^2}$$

THIN SLICE:



$$N = \underbrace{n_a}_{\text{DENSITY}} \overbrace{(A \Delta z)}^{\text{VOLUME}}$$

$$-\Delta I = \frac{P_{sc}}{A} = \frac{\hbar\omega R_{sc}}{A} = \frac{\hbar\omega\Gamma N \rho_{22}}{A}$$

$$= \hbar\omega\Gamma n_a \rho_{22} \Delta z$$

$$\frac{dI}{dz} = \frac{-\hbar\omega\Gamma n_a}{2 I_{SAT}} \underbrace{\frac{1}{1+S+\left(\frac{\delta}{\Gamma/2}\right)^2}}_{\text{DEPENDS ON } I} I \equiv - \underbrace{a}_{\text{ABSORPTION COEF.}} I$$

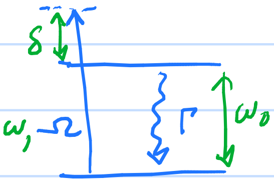
(FOOT USES κ)

$$[a] = \text{LENGTH}^{-1}$$

PHY 446 SPRING 2020
LECTURE 15

- ABSORPTION, SATURATION, & POWER BROADENING
- SCATTERING FORCE

TWO-LEVEL ATOM



RABI FREQ: $\Omega \propto E_0 \propto \sqrt{I}$

DECAY RATE: Γ

DETUNING: $\delta = \omega - \omega_0$

STEADY-STATE EXCITATION PROB.

$$\rho_{22}^{(ss)} = \frac{1}{2} \frac{s}{1+s+\left(\frac{2\delta}{\Gamma}\right)^2}$$

$$s = I/I_{SAT} = 2\Omega^2/\Gamma$$

← SATURATION INTENSITY

EXERCISE:

$\delta/\Gamma = 1$, $s = \frac{1}{2}$. FIND $\rho_{22}^{(ss)}$.

$$\rho_{22}^{(ss)} = \frac{1}{2} \frac{1/2}{1+\frac{1}{2}+2^2} = \frac{1}{4} \frac{1}{5.5} = \boxed{0.045}$$

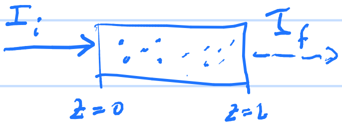
LAST TIME:

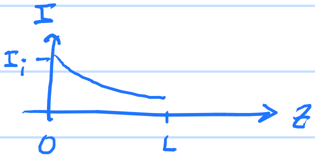
PHOTON SCATTERING RATE PER ATOM

$\Gamma \rho_{22}$

→ ABSORPTION COEFF.

ABSORPTION





$$\frac{dI}{dz} = -aI$$

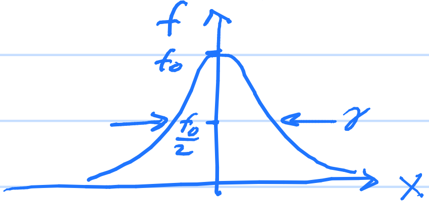
← ABSORPTION COEF.

$$a \approx \underbrace{\frac{\hbar\omega_0 \Gamma n_a}{2 I_{SAT}}}_{a_0} \frac{1}{1+S+\left(\frac{2\delta}{\Gamma}\right)^2} \equiv \frac{a_0}{1+S+\left(\frac{2\delta}{\Gamma}\right)^2}$$

LORENTZIAN FORM

GENERAL LORENTZIAN: $f(x) = \dfrac{f_0}{1+\left(\frac{x}{\gamma/2}\right)^2}$

f_0 = MAXIMUM, γ = FWHM



ABSORPTION COEF.: $$a = \frac{a_0/(1+S)}{1+\left[\frac{2\delta}{\Gamma\sqrt{1+S}}\right]^2}$$

MAXIMUM: $\dfrac{a_0}{1+S}$

- DECREASES WITH S ⇒ "SATURATION"

FWHM: $\Gamma\sqrt{1+S}$

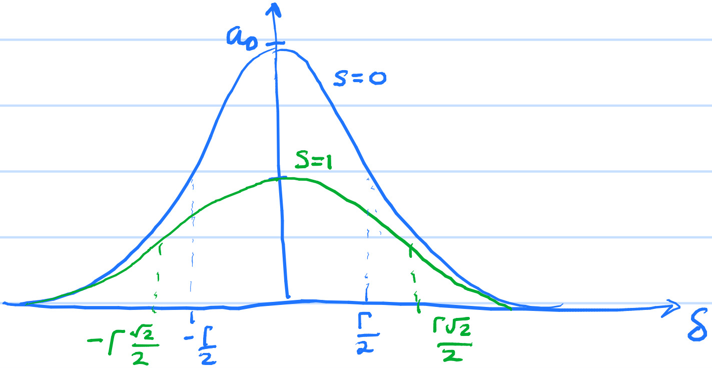
- INCREASES WITH S → "POWER BROADENING"

EXERCISE: FOR $s = 1$, FIND

a) MAX a $= a_0/(1+s) = \frac{1}{2} a_0$

b) FWHM(s) $= \Gamma\sqrt{1+s} = \Gamma\sqrt{2}$

PLOT: ABSORPTION vs. δ



SIMPLE FORMULA FOR a_0

SO FAR, $a_0 = \frac{\hbar\omega_0 \Gamma n_a}{2 I_{SAT}}$

FOR LIGHT W/ POLARIZATION $\hat{\varepsilon}$:

$$\frac{I_{SAT}}{\Gamma} = \frac{\hbar^2 \Gamma \varepsilon_0 c}{4 |d_{12}|^2} \quad ; \quad d_{12} = \langle 1 | \hat{\varepsilon} \cdot \vec{d} | 2 \rangle$$

GOOD NEWS: Γ IS RELATED TO $|d_{12}|^2$

FOR A TWO-LEVEL ATOM,

$$\frac{\Gamma}{|d_{12}|^2} = \frac{\omega_0^3}{3\pi \varepsilon_0 \hbar c^3}$$

- FROM QUANTUM THEORY OF LIGHT (WE'VE BEEN TREATING LIGHT AS A CLASSICAL FIELD)

COMBINE:

$$a_0 = \frac{\hbar\omega_0 n_a}{2}\left(\frac{4}{\hbar^2 \varepsilon_0 c}\right)\frac{3\pi\varepsilon_0 \hbar c^3}{\omega_0^3}$$

$$= 6\pi n_a \left(\frac{c}{\omega_0}\right)^2 \qquad = \boxed{6\pi n_a / k_0^2}$$

TWO-LEVEL ATOM ABSORPTION COEFFICIENT

$$a = \frac{6\pi n_a / k_0^2}{1 + s + (2\delta/\Gamma)^2}$$

COMPARE TO

LORENTZ OSCILLATOR RESULT ($|\delta| \ll \omega_0$)

$$\boxed{a = 2n_i k_0 \approx \frac{\omega_0}{c}\,\mathrm{Im}\chi} = \frac{\omega_0}{c}\frac{n_a}{\varepsilon_0}\,\mathrm{Im}\alpha$$

$$\alpha \approx \frac{-e^2}{2m\omega_0}\left(\frac{\delta - i\Gamma/2}{\delta^2+(\Gamma/2)^2}\right)$$

$$a \approx \frac{e^2}{2m\cancel{\omega_0}}\,\frac{\cancel{\omega_0}}{c}\,\frac{n_a}{\varepsilon_0}\,\frac{\Gamma/2}{(\Gamma/2)^2+\delta^2}$$

$$= \frac{e^2 n_a}{2\varepsilon_0 m_e c}\,\frac{2/\Gamma}{1+\left(\frac{\delta}{\Gamma/2}\right)^2} = \boxed{\frac{n_a e^2}{\Gamma\varepsilon_0 c m_e}\,\frac{1}{1+\left(\frac{\delta}{\Gamma/2}\right)^2}}$$

- SAME FORM IN $S\to 0$ LIMIT, $a \propto \frac{1}{1+\left(\frac{\delta}{\Gamma/2}\right)^2}$

SIMPLIFY USING CLASSICAL DECAY RATE

$$\Gamma_{cl} = \frac{e^2 k_0^2}{6\pi\varepsilon_0 m_e c}$$

$$a_{cl} = \left(\frac{6\pi\cancel{\varepsilon_0}\cancel{m_e}\cancel{c}}{\cancel{e^2}k_0^2}\right)\frac{\cancel{e^2}}{\cancel{\varepsilon_0}\cancel{m_e}\cancel{c}}\,n_a\,\frac{1}{1+(\delta/_{\Gamma/2})^2}$$

$$= \frac{6\pi}{k_0^2}\,n_a\,\frac{1}{1+\left(\frac{\delta}{\Gamma/2}\right)^2}$$

← MATCHES $S\to 0$ LIMIT

COMPARE TO 2-LEVEL MODEL

$$a_{\text{quantum}} = \frac{6\pi n_a/k_0^2}{1+S+(2\delta/\Gamma)^2}$$

↑ Quantum effect (saturation)

SCATTERING FORCE

$$F_{sc} = (\text{PHOTON MOMENTUM}) \times (\text{SCATTERING RATE})$$

$$= \hbar k \, \Gamma \rho_{22}$$

$$= \frac{\hbar k \Gamma}{2} \frac{S}{1+S+(2\delta/\Gamma)^2}$$

MAXIMUM ACCELERATION $(S \to \infty)$

$$a_{max} = \frac{F_{max}}{M} = \frac{\hbar k \Gamma}{2M}$$

Sodium: $\lambda = 589$ nm

$M = 23u$

$\Gamma = 9.8 \times 2\pi \times 10^6 /s$

$$a_{max} = 9 \times 10^5 \, m/s^2$$

CROSS SECTION

- COMMON WAY TO QUANTIFY ABSORPTION PER ATOM

DEFINE $\sigma \equiv \frac{a}{n_a}$ ← LENGTH^2

THEN $\frac{dI}{dz} = -\sigma n_a I$

$$\sigma = \frac{6\pi/k_0^2}{1+S+\left(\frac{\delta}{\Gamma/2}\right)^2} \equiv \frac{\sigma_0}{1+S+\left(\frac{\delta}{\Gamma/2}\right)^2}$$

WHERE $\sigma_0 = 6\pi/k_0^2$

$= \sigma(S=0, \delta=0)$

- RESONANT UNSATURATED ABS. CROSS SECTION